UNIVERSITÉ PARIS OUEST NANTERRE LA DÉFENSE

École doctorale **Connaissance, Langage, Modélisation** (ED 139)

Unité de recherche **Laboratoire MODAL'X**

Thèse présentée par **Julien Bureaux**

Soutenue le **10 décembre 2015**

En vue de l'obtention du grade de docteur de l'Université Paris Ouest Nanterre La Défense

Discipline **Mathématiques**

Titre de la thèse

Méthodes probabilistes pour l'étude asymptotique des partitions entières et de la géométrie convexe discrète

Thèse dirigée par Nathanaël Enriquez

Composition du jury

Rapporteurs	Philippe Marchal	chargé de recherche HDR
	Jean-François Marckert	directeur de recherche
Directeur de thèse	Nathanaël Enriquez	professeur

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Thesis defended by **Julien BUREAUX**

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Thesis Title

Probabilistic methods for the asymptotic study of integral partitions and discrete convex geometry

Thesis supervised by Nathanaël ENRIQUEZ

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Mots clés : partitions d'entiers multipartites, polygones convexes, polytopes à sommets entiers, zonotopes, théorème central limite local, combinatoire analytique

Keywords: partitions of multi-partite numbers, convex polygons, lattice polytopes, zonotopes, local central limit theorem, analytic combinatorics

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Méthodes probabilistes pour l'étude asymptotique des partitions entières et de la géométrie convexe discrète

Résumé

Cette thèse se compose de plusieurs travaux portant sur l'énumération et le comportement asymptotique de structures combinatoires apparentées aux partitions d'entiers.
Un premier travail s'intéresse aux partitions d'entiers bipartites, qui constituent une généralisation bidimensionnelle des partitions d'entiers. Des équivalents du nombre de partitions sont obtenus dans le régime critique où l'un des entiers est de l'ordre du carré de l'autre entier et au delà de ce régime critique. Ceci complète les résultats établis dans les années cinquante par Auluck, Nanda et Wright.
Le deuxième travail traite des chaînes polygonales à sommets entiers dans le plan. Pour un modèle statistique introduit par Sinaï, une représentation intégrale exacte de la fonction de partition est donnée. Ceci conduit à un équivalent du nombre de chaînes joignant deux points distants qui fait intervenir les zéros non triviaux de la fonction zêta de Riemann. Une analyse combinatoire détaillée des chaînes convexes est présentée. Elle permet de montrer l'existence d'une forme limite pour les chaînes convexes aléatoires ayant peu de sommets, répondant ainsi à une question ouverte de Vershik.
Un troisième travail porte sur les zonotopes à sommets entiers en dimension supérieure. Un équivalent simple est donné pour le logarithme du nombre de zonotopes contenus dans un cône convexe et dont les extrémités sont fixées. Une loi des grands nombres est établie et la forme limite est caractérisée par la transformée de Laplace du cône.

Mots clés : partitions d'entiers multipartites, polygones convexes, polytopes à sommets entiers, zonotopes, théorème central limite local, combinatoire analytique

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Probabilistic methods for the asymptotic study of integral partitions and discrete convex geometry

Abstract

This thesis consists of several works dealing with the enumeration and the asymptotic behaviour of combinatorial structures related to integer partitions.
A first work concerns partitions of large bipartite integers, which are a bidimensional generalization of integer partitions. Asymptotic formulæ are obtained in the critical regime where one of the numbers is of the order of magnitude of the square of the other number, and beyond this critical regime. This completes the results established in the fifties by Auluck, Nanda, and Wright.
The second work deals with lattice convex chains in the plane. In a statistical model introduced by Sinaï, an exact integral representation of the partition function is given. This leads to an asymptotic formula for the number of chains joining two distant points, which involves the non trivial zeros of the Riemann zeta function. A detailed combinatorial analysis of convex chains is presented. It makes it possible to prove the existence of a limit shape for random convex chains with few vertices, answering an open question of Vershik.
A third work focuses on lattice zonotopes in higher dimensions. An asymptotic equality is given for the logarithm of the number of zonotopes contained in a convex cone and such that the endings of the zonotope are fixed. A law of large numbers is established and the limit shape is characterized by the Laplace transform of the cone.

Keywords: partitions of multi-partite numbers, convex polygons, lattice polytopes, zonotopes, local central limit theorem, analytic combinatorics

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Introduction

1.1 Partitions entières

De combien de façons différentes peut-on écrire un nombre entier comme somme d'entiers naturels non nuls ? De façon équivalente, de combien de façons différentes peut-on partitionner un ensemble d'objets indiscernables ?

Si l'on considère égales deux écritures qui ne diffèrent que par l'ordre des termes de la somme, le nombre 6 admet par exemple onze écritures distinctes de ce type :

$$
\begin{aligned}
&6 = 6 && (0,0,0,0,0,1,0,\ldots)\\
&6 = 5+1 && (1,0,0,0,1,0,0,\ldots)\\
&6 = 4+2 && (0,1,0,1,0,0,0,\ldots)\\
&6 = 4+1+1 && (2,0,0,1,0,0,0,\ldots)\\
&6 = 3+3 && (0,0,2,0,0,0,0,\ldots)\\
&6 = 3+2+1 && (1,1,1,0,0,0,0,\ldots)\\
&6 = 3+1+1+1 && (3,0,1,0,0,0,0,\ldots)\\
&6 = 2+2+2 && (0,3,0,0,0,0,0,\ldots)\\
&6 = 2+2+1+1 && (2,2,0,0,0,0,0,\ldots)\\
&6 = 2+1+1+1+1 && (4,1,0,0,0,0,0,\ldots)\\
&6 = 1+1+1+1+1+1 && (6,0,0,0,0,0,0,\ldots)
\end{aligned}
$$

On peut formaliser ce type de décomposition avec la notion de multi-ensemble. Il sera cependant plus commode pour nous d'adopter le point de vue suivant qui lui est équivalent.

Définition. *On appelle* partition *d'un entier n, une application ω de $\mathbb{N}^*$ vers $\mathbb{N}$ telle que*

$$n = \omega(1) + 2\,\omega(2) + 3\,\omega(3) + \cdots$$

Pour tout entier n, on note $p(n)$ le nombre de partitions de n.

Les lecteurs auront remarqué que la somme qui intervient dans la définition n'est pas tout à fait de la même forme que les décompositions du nombre 6 que nous avons listées ci-dessus. La différence vient de ce qu'on regroupe les termes égaux, en notant $\omega(k)$ la multiplicité de chaque entier naturel k. Nous avons indiqué dans la colonne de droite la suite $(\omega(1), \omega(2), \omega(3), \omega(4), \ldots)$ qui correspond à chacune des onze partitions de 6.

La première apparition des partitions d'entiers dans la littérature mathématique semble être une lettre de Leibniz à Jacques Bernoulli datée de 1674. Leibniz remarque que $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$ sont tous des nombres premiers et il se demande si c'est un fait général. Le calcul de $p(7) = 15$ montre que ce n'est pas le cas, mais on ignore encore aujourd'hui s'il existe une infinité d'entiers n tels que $p(n)$ est premier.

1.1.1 Énumération des partitions d'entiers

De nombreuses questions naturelles se posent à propos des nombres $p(n)$. Ont-ils des propriétés arithmétiques particulières ? Y a-t-il des moyens efficaces de les calculer ? À quelle vitesse la suite $p(n)$ croît-elle ? En ce qui concerne la dernière de ces questions, on peut déjà remarquer que si ω est une partition de l'entier naturel n, alors on a nécessairement $k\,\omega(k) \leq n$ pour $1 \leq k \leq n$ et $\omega(k) = 0$ pour $k > n$, ce qui prouve le résultat suivant : pour tout entier naturel n, le nombre $p(n)$ de partitions de n est fini et vérifie

$$p(n) \leq \frac{n^n}{n!}.$$

L'équivalent de Stirling pour $n!$ montre alors que la croissance de $p(n)$ est au plus exponentielle en n lorsque n tend vers $+\infty$. On pourrait donc imaginer que $p(n)$ est équivalent à c^n pour une certaine constante c telle que $0 < c < e$, mais on verra un peu plus loin que l'ordre de grandeur correct est en fait $c^{\sqrt{n}}$.

Euler et les séries génératrices

C'est à Euler qu'on doit les premiers résultats vraiment profonds sur les partitions d'entiers, à la suite d'une lettre reçue en 1740 où Philippe Naudé le Jeune lui pose le problème suivant :

1. De combien de façons différentes le nombre 50 peut-il s'écrire comme la somme de 7 entiers naturels distincts ?
2. De combien de façons différentes le nombre 50 peut-il s'écrire comme la somme de 7 entiers naturels, égaux ou distincts ?

Euler présente sa solution [26] devant l'Académie des Sciences de Saint-Pétersbourg en 1741. Elle repose sur des techniques de manipulation de produits infinis et de séries infinies qu'il vient d'inventer et qui ne portent pas encore à cette époque le nom de *séries génératrices*. Il donnera en 1748 un exposé plus synthétique de ces techniques appliquées aux problèmes de partitions dans le chapitre XVI « De partitione numerorum » de son *Introductio in analysin infinitorum* [25].

La réponse à la première question consiste à observer que le nombre cherché est le coefficient devant le terme $x^{50}z^7$ dans le développement en série du produit infini

$$(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)\cdots$$

En langage moderne, les manipulations qu'Euler décrit ensuite reviennent à déduire de l'équation fonctionnelle $f(z)=(1+xz)f(xz)$, qui est vérifiée par ce produit, une relation de récurrence pour les coefficients $D(n,k)=D(n-k,k)+D(n-k,k-1)$ à partir de laquelle il calcule $D(50,7)$ de proche en proche.

La deuxième question se traite de façon similaire en considérant cette fois le produit infini

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)\cdots}$$

À la suite de ce problème de Naudé, Euler étend naturellement le problème à la question du calcul des nombres $p(n)$.

Théorème. *Pour tout entier naturel n, le nombre $p(n)$ est le coefficient de z^n dans le développement en série de*

$$\prod_{k=1}^{\infty}\frac{1}{1-z^k}.$$

La formule de Hardy et Ramanujan

Il faudra attendre cent-cinquante ans après Euler pour que débute l'étude asymptotique de la suite $p(n)$ avec les premiers échanges épistolaires entre Ramanujan et Hardy dès 1913. Dans l'un des plus célèbres articles [31] de leur collaboration, ils prouvent que

$$\log p(n) \underset{n\to\infty}{\sim} \pi\sqrt{\frac{2n}{3}}$$

à l'aide d'un théorème taubérien. Puis ils introduisent ce qui est maintenant connu sous le nom de *méthode du cercle* afin de déterminer un développement asymptotique complet de $p(n)$, le premier terme duquel fournit la remarquable formule suivante.

Théorème (Hardy, Ramanujan)**.**

$$p(n) \underset{n\to\infty}{\sim} \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Notons que Rademacher [40] prolongera les résultats de Hardy et Ramanujan en 1937. Il aboutira à une expression de $p(n)$ comme somme d'une série convergente :

Théorème (Rademacher)**.**

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{\frac{1}{2}}\right)}{\left(x - \frac{1}{24}\right)^{\frac{1}{2}}} \right]_{x=n}$$

avec

$$A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i s(h,k)}$$

et

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \left\lfloor \frac{\mu}{k} \right\rfloor - \frac{1}{2} \right) \left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2} \right).$$

Les méthodes développées par Hardy et Ramanujan ont eu un impact considérable en théorie des nombres tout au long du xx^{e} siècle, apportant des résultats profonds sur le problème de Waring, les nombres premiers jumeaux, la conjecture de Goldbach, etc. Ils font cependant la remarque suivante dans leur article de 1918.

> « Il ne fait aucun doute que des arguments « taubériens »permettent de prouver bien mieux qu'un équivalent de $\log p(n)$. Mais il semble peu probable que des arguments de cette nature puissent nous amener à une preuve de l'équivalent de $p(n)$. Il serait excessivement intéressant de savoir jusqu'où de tels arguments peuvent nous porter, puisque cette méthode est relativement élémentaire et possède un spectre d'application bien plus large que les méthodes plus puissantes employées dans cet article. »

Ingham [32] montre en 1941 que des arguments de nature taubérienne sont en fait suffisants pour établir l'équivalent de $p(n)$, et Meinardus [35] établit en 1954 un théorème qui fournit des équivalents du même type pour une large classe de partitions. Newman [37] donne en 1962 une preuve particulièrement courte de la formule de Hardy et Ramanujan. Tous ces résultats reposent essentiellement sur l'estimation d'intégrales de contour complexes à l'aide de la méthode du col.

Erdős [22] parvient pourtant en 1942 à démontrer l'équivalent à une constante multiplicative près de façon purement élémentaire, sans théorie taubérienne ni analyse complexe. Newman [38] complètera ce travail en 1951 en obtenant la constante.

Nous ne résistons pas à l'envie d'esquisser la preuve d'Erdős pour l'équivalent logarithmique. Le point de départ de son approche est l'identité

$$n\,p(n) = \sum_{v \geq 1} \sum_{k \geq 1} v\,p(n - kv)$$

où la somme est prise sur l'ensemble des couples (v,k) tels que $kv \leq n$. Cette identité s'obtient facilement en considérant la dérivée logarithmique de la fonction génératrice des $p(n)$. Mais on peut aussi l'interpréter de façon combinatoire en remarquant que parmi les $p(n)$ partitions ω de n, il y en a exactement $p(n-kv)$ pour lesquelles $\omega(v) \geq k$. Dans la partition de $n\,p(n)$ composée de la réunion des partitions de n, la multiplicité du terme v est donc $\sum_{k\geq 1} p(n-kv)$, d'où l'identité.

À partir d'ici, la preuve de la majoration $p(n) \leq \exp\left(\pi\sqrt{2n/3}\right)$ se fait simplement par récurrence sur n et on peut prouver de la même manière une minoration de la forme $p(n) \geq \frac{1}{A}\exp\left(c\sqrt{n}\right)$ pour tout $c < \pi\sqrt{2/3}$.

Nous verrons plus loin qu'une nouvelle démonstration de la formule de Hardy et Ramanujan reposant sur des arguments probabilistes a été donnée par Báez-Duarte [7] en 1997.

1.1.2 Comportement asymptotique des partitions aléatoires

Approche élémentaire

La structure de l'ensemble des partitions d'un entier n est relativement complexe. Mais si on adopte un point de vue probabiliste, on réussit à dégager des comportements *en loi* dans le régime asymptotique. Erdős et Lehner [23] sont les premiers à se lancer dans une telle étude. Ils s'intéressent plus spécifiquement au nombre de parts d'une partition tirée uniformément parmi les partitions de n.

Théorème (Erdős, Lehner). *La loi du nombre de parts K_n d'une partition aléatoire de n tirée uniformément converge, après une normalisation convenable, vers une loi de Gumbel. Pour tout x réel,*

$$\Pr\left(\frac{\pi}{\sqrt{6n}}K_n - \log\frac{\sqrt{6n}}{\pi} \leq x\right) \xrightarrow[n\to\infty]{} \exp\left(-e^{-x}\right).$$

L'apparition d'une loi d'extremum comme la loi de Gumbel n'est pas trop étonnante si on remarque que la loi du nombre de parts d'une partition aléatoire coïncide avec la loi de la plus grande part de la partition, ce qui résulte de la dualité des diagrammes de Ferrer.

D'autres fonctionnelles naturelles des partitions aléatoires d'entiers seront considérées ultérieurement par l'école hongroise [48, 47, 24] en lien avec l'étude statistique du groupe symétrique $\mathfrak{S}_n$.

Point de vue probabiliste

Une avancée importante est réalisée en 1993 par Fristedt [28] qui propose une nouvelle approche, de nature plus probabiliste. Grâce à celle-ci, il complète et généralise les résultats d'Erdős, Turán et Szalay de façon remarquablement efficace. Le point crucial de sa méthode est l'observation suivante : la loi des multiplicités d'une patition aléatoire de n est égale à la loi d'une famille de variables aléatoires indépendantes, conditionnée par la valeur de la somme de toutes les parts (qui doit valoir n). Ceci permet d'approcher la loi de la partition de n par la loi d'une partition d'un entier aléatoire N d'espérance n dont les multiplicités des parts sont indépendantes. La qualité de l'approximation est quantifiée par la formule de Hardy et Ramanujan [31]. Notons que l'approche de Fristedt s'inscrit en fait dans une classe beaucoup plus large de structures combinatoires aléatoires dont Arratia et Tavaré [5] donnent un panorama. Cette classe comprend notamment les permutations aléatoires, les fonctions aléatoires et les partitions aléatoires d'un ensemble fini.

Les idées de Fristedt sont reprises par la suite pour étudier de nouvelles fonctionnelles des partitions aléatoires d'entiers. Citons en particulier le travail de Goh et Schmutz [29] qui établissent un théorème central limite pour le nombre de tailles distinctes parmi les parts d'une partition aléatoire, et le travail de Corteel, Pittel, Savage et Wilf [20] sur la multiplicité d'une part prise au hasard dans une partition aléatoire.

Alors que l'article de Fristedt repose sur l'équivalent de Hardy et Ramanujan [31] pour établir le lien entre le modèle uniforme et le modèle approché, Báez-Duarte [7] observe qu'on peut en fait procéder dans l'autre sens. Avec le point de vue probabiliste, la formule de Hardy et Ramanujan s'interprète comme une version *locale* du théorème central limite qui peut être démontrée en utilisant les outils usuels de la théorie des probabilités : moments, fonction caractéristique. Cette observation joue un rôle fondamental dans notre approche du problème analogue en dimension supérieure.

Nous nous proposons ici d'illustrer les approches de Fristedt et Báez-Duarte en donnant une deuxième preuve de l'équivalent $\log p(n) \sim \pi\sqrt{2n/3}$. On introduit une famille $(\omega(k))_{k\geq 1}$ de variables aléatoires indépendantes de loi géométrique

$$\mathbb{P}_\beta(\omega(k) = j) = (1 - e^{-\beta k})e^{-\beta k j}, \qquad j \geq 0$$

dépendant d'un paramètre $\beta > 0$. On introduit la variable aléatoire

$$N = \sum_{k\geq 1} \omega(k)k$$

dont l'espérance est égale à

$$\sum_{k\geq 1} \frac{ke^{-\beta k}}{1 - e^{-\beta k}} = \sum_{r\geq 1} \frac{e^{-\beta r}}{(1 - e^{-\beta r})^2} \underset{\beta\to 0}{\sim} \frac{\zeta(2)}{\beta^2}.$$

Notons que l'existence de $\mathbb{E}(N) < \infty$ entraîne que $\mathbb{P}(N < \infty) = 1$. La suite $(\omega(k))_{k\geq 1}$ n'a donc presque sûrement qu'un nombre fini de termes non nuls, quel que soit $\beta > 0$. La fonction $\beta \mapsto \mathbb{E}_\beta(N)$ définit par ailleurs un homéomorphisme décroissant de $]0, \infty[$ dans lui-même. Ceci va nous permettre de calibrer β en fonction de $\mathbb{E}(N)$.

Nous allons déduire l'équivalent logarithmique de $p(n)$ du comportement asymptotique de $\mathbb{P}_\beta(N = n)$ pour un choix adapté de β en fonction de n. Commençons par observer que l'ensemble des suites ω telles que $N(\omega) = n$ est exactement l'ensemble des partitions de n, par définition de ce dernier. De plus, la probabilité de chacune des suites ω telles que $N(\omega) = n$ est égale à

$$\prod_{k\geq 1} (1 - e^{-\beta k})e^{-\beta k\omega(k)} = \frac{e^{-\beta n}}{Z(\beta)}, \qquad \text{avec } Z(\beta) = \prod_{k\geq 1} \frac{1}{1 - e^{-\beta k}}.$$

On en déduit notamment que

$$\frac{p(n)e^{-\beta n}}{Z(\beta)} = \mathbb{P}_\beta(N = n) \leq 1,$$

d'où $\log p(n) \leq \beta n + \log Z(\beta)$. Si l'on choisit maintenant $\beta > 0$ en fonction de n de telle sorte que $\mathbb{E}_\beta(N) = n$, l'équivalent de $\mathbb{E}_\beta(N)$ pour $\beta \to 0$ entraîne que

$$\beta \sim \sqrt{\frac{\zeta(2)}{n}}.$$

Puisqu'on a par ailleurs un équivalent

$$\log Z(\beta) = \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{-\beta r}}{1 - e^{-\beta r}} \sim \frac{\zeta(2)}{\beta} \sim \sqrt{\zeta(2)n},$$

la majoration obtenue ci-dessus conduit à

$$\limsup_{n\to\infty} \frac{\log p(n)}{\sqrt{n}} \leq \pi\sqrt{\frac{2}{3}}.$$

Pour obtenir une minoration, on va considérer le moment d'ordre 2. On choisit cette fois β de sorte que $\mathbb{E}_\beta(N) = n - n^\kappa$ avec $3/4 < \kappa < 1$, ce qui ne change pas l'équivalent de β lorsque $n \to \infty$. On montre facilement que la variance de N est de l'ordre de $n^{3/2}$. L'inégalité de Bienaymé-Tchebychev entraîne alors que

$$\mathbb{P}(n - 2n^\kappa \leq N \leq n) \xrightarrow[n\to\infty]{} 1.$$

Mais la croissance de la suite $(p(k))_{k\in\mathbb{N}}$ montre que

$$\mathbb{P}(n - 2n^\kappa \leq N \leq n) = \sum_{n-2n^\kappa \leq k \leq n} \frac{p(k)e^{-\beta k}}{Z(\beta)} \leq \frac{p(n)e^{-\beta n}}{Z(\beta)} \times (1 + 2n^\kappa)e^{2\beta n^\kappa}.$$

En passant au logarithme, on obtient finalement (puisque $\kappa < 1$)

$$\liminf_{n\to\infty} \frac{\log p(n)}{\sqrt{n}} \geq \pi\sqrt{\frac{2}{3}}.$$

Forme limite

Au delà de l'existence de lois limites pour diverses fonctionnelles ponctuelles des partitions aléatoires, une question naturelle consiste à se demander s'il est également possible d'observer un comportement limite global. Pour toute partition ω de n, notons $\lambda_k(\omega)$ la longueur de la k-ième plus grande part de la partition ω, c'est-à-dire l'unique entier vérifiant les relations

$$\sum_{i \geq \lambda_k(\omega)} \omega(i) \geq k, \qquad \sum_{i > \lambda_k(\omega)} \omega(i) < k.$$

Vershik [54, 52] remarque que d'après des bornes de Szalay et Turán [47], les longueurs λ_k d'une partition de n tirée uniformément vérifient grossièrement l'équation déterministe

$$\exp\left(-\frac{\pi k}{\sqrt{6n}}\right) + \exp\left(-\frac{\pi \lambda_k}{\sqrt{6n}}\right) = 1$$

avec une forte probabilité.

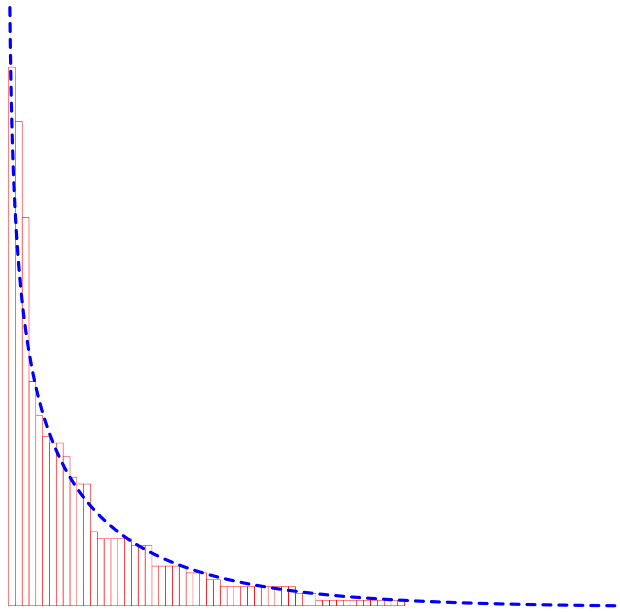

FIGURE 1.1 – Forme limite de Vershik et Temperley

Théorème (Vershik)**.** *Soit Γ la courbe d'équation cartésienne $e^{-x} + e^{-y} = 1$. Pour toute partition ω de l'entier n, soit $E_n(\omega)$ l'ensemble formé des points $(\frac{\pi k}{\sqrt{6n}}, \frac{\pi \lambda_k}{\sqrt{6n}})$ pour $k \geq 1$. Lorsque l'ensemble des partitions de n est muni de la mesure uniforme, on a la convergence en probabilité*

$$\sup_{p \in E_n} d(p, \Gamma) \xrightarrow[n\to\infty]{} 0.$$

En utilisant les idées de Fristedt, Pittel [39] fournit une démonstration détaillée de cette loi des grands nombres, et va plus loin en établissant un théorème central limite fonctionnel. Il montre que le processus des fluctuations est gaussien (mais pas markovien) et il explicite la fonction de covariance.

La courbe Γ apparaît en fait dès 1952 dans un article de Temperley [49]. Il l'obtient par des arguments heuristiques de physique statistique pour un processus de partitions d'entiers qui modélise la croissance d'un cristal.

En 2001, Vershik et Yakubovich [55] ont explicité de nouvelles formes limites pour les partitions tirées uniformément mais conditionnées par leur nombre de parts. Ils obtiennent ainsi un continuum de courbes autour desquelles ils démontrent l'existence de fluctuations gaussiennes.

1.1.3 Partitions d'entiers multipartites

En dimension $d \geq 1$, de combien de façons différentes peut-on écrire un vecteur à coefficients entiers comme somme de vecteurs non nuls dont les composantes sont des entiers naturels ? De façon équivalente, de combien de façons différentes est-t-il possible de partitionner un ensemble d'objets indiscernables sauf par leur couleur (choisie parmi d possibilités) ?

Dans le cas monochrome $d = 1$, il s'agit du problème des partitions d'un entier que nous venons d'aborder. Dans le cas $d \geq 1$, nous avons affaire à une généralisation non triviale dont l'étude a été amorcée au milieu du xx^e siècle par de nombreux auteurs avec des approches variées.

L'une des motivations principales provient de la physique statistique. Au début des années 1950, Fermi [27] a en effet introduit un modèle thermodynamique dans lequel l'état du système est caractérisé par la conservation de deux « énergies » plutôt qu'une. Ces travaux ont conduit Auluck [6] à étudier le nombre de partitions des entiers bipartis, c'est-à-dire le nombre de partitions d'un vecteur (n_1, n_2) à composantes entières. Il parvient à établir des équivalents à la Hardy-Ramanujan dans deux régimes asymptotiques bien distincts. Lorsque n_1 est fixé et n_2 tend vers $+\infty$, le nombre de partitions de (n_1, n_2) est équivalent à

$$\frac{1}{n_1!}\left(\frac{\sqrt{6n_2}}{\pi}\right)^{n_1}\frac{1}{4n_2\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n_2}{3}}\right).$$

La deuxième situation considérée par Auluck est celle où n_1 et n_2 tendent tous les deux vers $+\infty$ en restant du même ordre de grandeur. La formule qu'il obtient dans ce cas est beaucoup plus complexe. Aussi nous nous contenterons de remarquer que lorsque $n_1 = n_2 = n$, elle prend la forme

$$\frac{a}{n^{\frac{55}{36}}}\exp\left(bn^{\frac{2}{3}} + cn^{\frac{1}{3}}\right)$$

où les nombres a, b, c sont des constantes explicites définies à partir des fonctions spéciales.

Alors que cette deuxième formule fait apparaître des constantes et des exposants inattendus, la première a une interprétation combinatoire intuitivement simple. Sachant que le nombre typique de parts d'une partition de n est $\pi^{-1}\sqrt{6n}$, le choix d'une partition de (n_1, n_2), lorsque n_1 est fixé, se décompose approximativement en :

- le choix d'une partition de l'entier n_2, ce qui fait apparaître l'équivalent de Hardy et Ramanujan
$$\frac{1}{4n_2\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n_2}{3}}\right);$$
- le choix de n_1 parts parmi celles de la partition de n_2 pour lesquelles la première composante sera non nulle, ce qui fait apparaître le coefficient binomial
$$\binom{\sqrt{6n_2/\pi}}{n_1} \sim \frac{1}{n_1!}\left(\frac{\sqrt{6n_2}}{\pi}\right)^{n_1}.$$

Nanda [36] montre en 1957 que cette situation perdure tant que n_1 reste suffisamment petit devant n_2, plus précisément tant que $n_1 = o(n_2^{1/4})$. Sa démonstration consiste essentiellement à rendre rigoureuse la décomposition heuristique que nous venons de formuler.

Parallèlement, Wright [57] parvient à prouver que le comportement asymptotique correspondant à la deuxième formule d'Auluck s'étend en fait au vaste domaine défini par les conditions
$$\frac{1}{2} < \liminf \frac{n_2}{n_1} \leq \limsup \frac{n_2}{n_1} < 2.$$
Enfin, les travaux de Robertson [41, 42] au début des années 1960 généralisent les résultats précédents en dimension d quelconque.

Les articles d'Aucluck, Nanda, Wright et Robertson reposent tous sur l'utilisation de fonctions génératrices bivariées. À l'exception de Nanda dont la preuve reste élémentaire, la difficulté technique majeure de ces travaux consiste à estimer le plus précisément possible la fonction génératrice, puis à extraire le comportement asymptotique de ses coefficients à l'aide de théorèmes taubériens ou de la méthode du col.

1.1.4 Notre contribution

Nous nous sommes intéressé au comportement asymptotique du nombre de partitions du vecteur (n_1, n_2) dans le régime critique $n_1 \asymp \sqrt{n_2}$ et dans le régime sous-critique $n_1 = o(\sqrt{n_2})$. Cette étude complète les résultats établis dans les années 1950 par Auluck, Nanda et Wright.

Les recherches présentées dans cette section ont fait l'objet d'une publication dans les *Proceedings of the Cambridge Philosophical Society* [19].

Partitions d'entiers bipartis déséquilibrés

Notons respectivement $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$ et $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$ l'ensemble des entiers naturel et l'ensemble des entiers naturels non nuls. Nous rappelons que les notations asymptotiques de Landau $a_n = o(b_n)$ or $a_n = O(b_n)$ pour deux suites de réels à termes non nuls (a_n) et (b_n) signifient respectivement $\limsup \left|\frac{a_n}{b_n}\right| = 0$ et $\limsup \left|\frac{a_n}{b_n}\right| < +\infty$. Nous utiliserons également la notation $a_n \asymp b_n$ pour indiquer que a_n and b_n ont asymptotiquement le même ordre de grandeur, c'est-à-dire que les deux conditions $a_n = O(b_n)$ et $b_n = O(a_n)$ sont satisfaites.

Définition. *Soit X un sous-ensemble de $\mathbb{Z}_+^2$. Pour tout $n \in \mathbb{Z}_+^2$, une* partition *de n à* parts *dans X est une famille non-ordonnée finie d'éléments de X dont la somme vaut n. On peut la décrire par sa fonction de* multiplicité *$\omega\colon X \to \mathbb{Z}_+$ qui vérifie la condition $\sum_{x\in X} \omega(x) \cdot x = n$. Pour chaque $x \in X$, nous dirons que $\omega(x)$ est la multiplicité de la part x dans la partition ω. Les partitions de n à parts dans X constituent l'ensemble*

$$\Omega_X(n) := \left\{ \omega \in \mathbb{Z}_+^X : \sum_{x\in X} \omega(x) \cdot x = n \right\}.$$

Enfin, nous noterons $p_X(n) := |\Omega_X(n)|$ le nombre de partitions de n à parts dans X.

De façon similaire aux travaux de Wright and Robertson [57, 41, 42], nous nous concentrerons essentiellement sur les deux choix suivants pour l'ensemble des parts X dans ce chapitre, à savoir :

— $X = \mathbb{N}^2$, qui correspond aux partitions dont aucune part n'a de composante nulle.

— $X = \mathbb{Z}_+^2 \setminus \{0\}$, qui correspond au cas plus général de partitions, dans lequel les parts peuvent avoir une composante nulle. Nous devons encore exclure la possibilité d'une part nulle de façon à nous assurer que chaque vecteur n'a qu'un nombre fini de partitions.

Le théorème suivant résume nos principaux résultats sur le problème de l'énumération asymptotique des partitions. Il décrit le comportement asymptotique de $p_X(n)$ dans le cas des partitions sans composante nulle ainsi que dans le cas général, dans le complémentaire de la région étudiée par Wright. Avant de l'énoncer, nous avons besoin d'introduire quelques fonctions auxiliaires de la variable $\alpha > 0$:

$$\Phi(\alpha) = \sum_{r\geq 1} \frac{1}{r^2} \frac{e^{-\alpha r}}{1-e^{-\alpha r}}, \qquad \overline{\Phi}(\alpha) = \Phi(\alpha) + \frac{\pi^2}{6},$$

$$\Theta(\alpha) = -\frac{\Phi'(\alpha)}{\sqrt{\Phi(\alpha)}}, \qquad \overline{\Theta}(\alpha) = -\frac{\overline{\Phi}'(\alpha)}{\sqrt{\overline{\Phi}(\alpha)}},$$

$$\Delta(\alpha) = 2\Phi(\alpha)\Phi''(\alpha) - \Phi'(\alpha)^2, \qquad \overline{\Delta}(\alpha) = 2\overline{\Phi}(\alpha)\overline{\Phi}''(\alpha) - \overline{\Phi}'(\alpha)^2,$$

$$\Psi(\alpha) = \sum_{r\geq 1} \frac{1}{r} \frac{e^{-\alpha r}}{1-e^{-\alpha r}}.$$

Considérons deux suite $(n_1(k))_k$ et $(n_2(k))_k$ d'entiers naturels. Dans le reste de ce chapitre, les passages à limite et les relations asymptotiques seront pris, sauf indication contraire, pour k tendant vers l'infini. Cependant, nous laisserons désormais l'indice k implicite.

Théorème. *Supposons que les entiers n_1 et n_2 tendent tous les deux vers l'infini et vérifient les conditions $n_1 = O(\sqrt{n_2})$ et $\log(n_2) = o(n_1)$.*

(i) L'équation $\Theta(\alpha_n) = \dfrac{n_1}{\sqrt{n_2}}$ a une unique solution $\alpha_n > 0$, et $p_{\mathbb{N}^2}(n_1, n_2)$ est asymptotiquement équivalent à

$$\frac{1}{2\pi}\frac{\Phi(\alpha_n)}{n_2}\frac{e^{-\frac{1}{2}\Psi(\alpha_n)}}{\sqrt{\Delta(\alpha_n)}}\exp\left\{\left(\alpha_n\Theta(\alpha_n)+2\sqrt{\Phi(\alpha_n)}\right)\sqrt{n_2}\right\}.$$

(ii) L'équation $\overline{\Theta}(\alpha_n) = \dfrac{n_1}{\sqrt{n_2}}$ a une unique solution $\alpha_n > 0$, et $p_{\mathbb{Z}_+^2\setminus\{0\}}(n_1, n_2)$ est asymptotiquement équivalent à

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{\overline{\Phi}(\alpha_n)}{n_2}\right)^{\frac{5}{4}}\frac{e^{\frac{1}{2}\Psi(\alpha_n)}}{\sqrt{\overline{\Delta}(\alpha_n)}}\exp\left\{\left(\alpha_n\overline{\Theta}(\alpha_n)+2\sqrt{\overline{\Phi}(\alpha_n)}\right)\sqrt{n_2}\right\}.$$

Bien que les formules du théorème fassent intervenir une fonction implicite α_n de (n_1, n_2), remarquons qu'elles permettent en fait d'obtenir des développements asymptotiques explicites en termes de n_1 et n_2 lorsque n_1 est négligeable devant $\sqrt{n_2}$, condition qui équivaut à $\alpha_n \to +\infty$. Ceci vient du fait que les fonctions auxiliaires $\Phi, \overline{\Phi}, \Theta, \overline{\Theta}, \Delta, \overline{\Delta}$, et Ψ admettent des développements asymptotiques simples lorsque $\alpha \to +\infty$ s'exprimant à l'aide de la fonction arithmétique « somme des carrés diviseurs »

$$\sigma_2(m) = \sum_{d|m} d^2.$$

Ceci découle directement des identités élémentaires suivantes pour les séries de Lambert [51, Section 4.71] :

$$\Phi(\alpha) = \sum_{m\geq 1}\frac{\sigma_2(m)}{m^2}e^{-\alpha m}, \qquad -\Phi'(\alpha) = \sum_{m\geq 1}\frac{\sigma_2(m)}{m}e^{-\alpha m}.$$

Les développements asymptotiques de $\Theta^{-1}, \overline{\Theta}^{-1}$ peuvent être calculés effectivement à partir de ces expressions par un procédé itératif ou en utilisant la formule de réversion de Lagrange. Illustrons maintenant comment ces idées simples nous permettent d'étendre les résultats de Nanda and Robertson.

Pour un premier exemple, considérons le cas (i). L'application du théorème fournit des termes correctifs additionnels dans le développement donné par Robertson [41, Theorem 2].

Corollaire. *Il existe une suite* (c_k) *de nombres rationnels tels que pour tout* $K \in \mathbb{N}$*, si* n_1 *et* n_2 *tendent vers* $+\infty$ *et vérifient les conditions* $n_1^{2K+1} = o(n_2^K)$ *et* $\log(n_2) = o(n_1)$*, alors*

$$p_{\mathbb{N}^2}(n_1, n_2) \sim \frac{n_1}{n_2} \frac{n_2^{n_1}}{(n_1!)^2} \exp\left\{ \sum_{k=1}^{K-1} c_k \frac{n_1^{2k+1}}{n_2^k} \right\}.$$

L'énoncé de Robertson ne considère que le cas particulier $K = 1$, correspondant au régime $n_1 = o(n_2^{1/3})$.

La démonstration de cette formule, qui sera donnée dans un chapitre ultérieur, se traduit directement par un algorithme effectif de calcul de la suite. Voici les cinq premiers termes de la suite (c_k) que nous avons calculé à l'aide du logiciel de calcul formel *Sage* [45] :

$$c_1 = \frac{5}{4},\ c_2 = -\frac{805}{288},\ c_3 = \frac{6731}{576},\ c_4 = -\frac{133046081}{2073600},\ c_5 = \frac{170097821}{414720}.$$

Considérons maintenant le cas (ii). De manière analogue au cas précédent, l'application du théorème conduit à une extension de la formule d'Auluck et Nanda.

Corollaire. *Il existe une suite* $(\bar{c}_k)$ *de nombres rationnels tels que pour tout* $K \in \mathbb{N}$*, si* n_1 *et* n_2 *tendent vers* $+\infty$ *et vérifient les conditions* $n_1^{K+1} = o(n_2^{K/2})$ *et* $\log(n_2) = o(n_1)$*, alors*

$$p_{\mathbb{Z}_+^2}(n_1, n_2) \sim \frac{1}{n_1!} \left(\frac{\sqrt{6n_2}}{\pi} \right)^{n_1} \frac{1}{4n_2\sqrt{3}} \exp\left\{ \pi \sqrt{\frac{2n_2}{3}} \right\} \exp\left\{ \sum_{k=1}^{K-1} \bar{c}_k \frac{n_1^{k+1}}{n_2^{k/2}} \right\}.$$

Seul le cas $n_1 = o(n_2^{1/4})$, qui correspond à $K = 1$, est présent dans le travail de Nanda [36].

En posant $a = \sqrt{\zeta(2)}$, le calcul effectif des premiers termes de la suite $(\bar{c}_k)$ donne

$$\bar{c}_1 = \frac{5a}{4} - \frac{1}{4a},\ \bar{c}_2 = \frac{5}{8} - \frac{145a^2}{72},\ \bar{c}_3 = 6a^3 - \frac{1385a}{576} + \frac{5}{32a} + \frac{1}{192a^3}.$$

Remarquons que les développements asymptotiques fournis par ces deux corollaires sont naturellement limités lorsqu'on fait croître K par le régime critique $n_1 \asymp \sqrt{n_2}$. Dans ce cas critique, par exemple lorsque $n_1/\sqrt{n_2}$ converge suffisamment vite vers une constante non nulle, tous les termes dépendant de α_n tendent vers des coefficients indépendants de n_1 et n_2. C'est le cas notamment dans le cas naturel $n_1 = n$ et $n = n^2$.

Plus généralement, le théorème donne l'existence ainsi qu'une formule (implicite) pour les exposants limites

$$h(t) := \lim_{n\to+\infty} \frac{1}{\sqrt{n}} \log p_{\mathbb{N}^2}(\lfloor t\sqrt{n} \rfloor, n), \quad \bar{h}(t) = \lim_{n\to+\infty} \frac{1}{\sqrt{n}} \log p_{\mathbb{Z}_+^2\setminus\{0\}}(\lfloor t\sqrt{n} \rfloor, n),$$

qui sont définis pour tout $t > 0$.

Une nouvelle approche

La méthode que nous employons diffère sensiblement de l'approche purement analytique d'Auluck [6], Wright [57] et Robertson [41, 42]. Elle repose en effet de façon importante sur l'introduction d'un modèle probabiliste du type *ensemble grand canonique*, inspiré du modèle de Boltzmann en physique statistique. Ce point de vue a l'avantage de permettre la mise en place de techniques systématiques. Il fournit également un modèle simplifié de partitions aléatoires dont le comportement est très proche de celui des partitions tirées uniformément. On peut considérer cette approche comme une généralisation des idées développée par Fristedt [28] dans l'étude de la structure des partitions aléatoires des grands entiers, combinée à la démonstration de Báez-Duarte [7] de la formule de Hardy et Ramanujan.

La première partie de la méthode consiste alors à estimer précisément la fonction de partition logarithmique du modèle statistique considéré. Ceci est rendu possible par une représentation exacte de cette fonction sous forme d'intégrale de contour complexe. L'approximation découle alors d'une application du théorème des résidus de Cauchy. L'estimation de la fonction de partition donne accès à des théorèmes limites pour le modèle aléatoire simplifié.

La deuxième partie de la méthode permet de transporter les résultats obtenus pour ce modèle simplifié au véritable modèle de partitions aléatoires tirées uniformément. Il s'agit pour cela d'établir une version locale du théorème central limite dans $\mathbb{Z}^2$. Cet argument joue un rôle crucial dans le cadre de la physique statistique où il fournit une justification rigoureuse du principe d'*équivalence des ensembles*. Compte tenu de ce caractère relativement universel, nous avons introduit un cadre général qui permet de réduire la preuve du théorème central limite local à la vérification de trois conditions sur le modèle.

Sinaĭ [44] semble avoir été le premier à utiliser une méthode de ce type dans un contexte multi-dimensionnel. Il s'agissait d'un problème voisin du nôtre, sur lequel nous reviendrons dans la prochaine section. Vershik [52] présente une discussion générale sur ces techniques et un article récent de Bogachev et Zarbaliev [17] fournit, parmi d'autres résultats, une démonstration détaillée de l'approche de Sinaĭ. Notons que les théorèmes limites pour les modèles concernés par ces travaux tombent dans notre cadre général. Mentionnons enfin que le problème des partitions d'entiers bipartis déséquilibrés présente une forte anisotropie spatiale qui rend la mise en place de ce programme plus délicate que dans les articles précités.

1.2 Polygones convexes à sommets entiers

L'énumération des points à coordonnées entières sur des courbes ou dans des domaines constitue l'un des thèmes principaux de la théorie des nombres [30]. Le problème suivant, étudié par Jarník [33] en 1926, en est un archétype. Soit γ une courbe fermée strictement convexe de longueur $\ell \geq 1$ tracée dans le plan $\mathbb{R}^2$. Par combien de points à

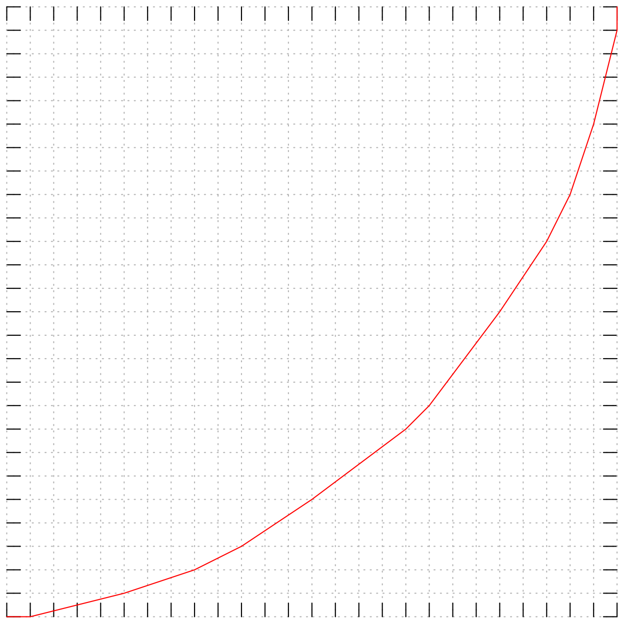

Figure 1.2 – Quart sud-est du polygone de Jarník

coordonnées entières la courbe γ passe-t-elle ? Jarník montre que ce nombre est au plus

$$\frac{3}{(2\pi)^{\frac{1}{3}}}\ell^{\frac{2}{3}} + O(\ell^{\frac{1}{3}})$$

et que cette majoration est la meilleure possible. Sa preuve repose sur la construction, pour un nombre de sommets fixé, d'un polygone convexe à sommets entiers de périmètre minimal. Les côtés du polygone de Jarník sont formés des vecteurs $(a,b) \in \mathbb{Z}^2$ avec a et b premiers entre eux tels que $a^2 + b^2 \leq t$, où le seuil $t \geq 1$ est choisi en fonction du nombre de sommets à atteindre.

Cette question s'est révélée extrêmement féconde et a conduit à de nombreux prolongements en analyse diophantienne. Citons notamment les travaux de Swinnerton-Dyer [46], Schmidt [43], Bombieri et Pila [18] qui considèrent le cas de courbes plus régulières.

Le problème de Jarník se généralise naturellement en dimension supérieure : comment peut-on majorer le nombre de points entiers à la surface d'un convexe compact de l'espace euclidien $\mathbb{R}^d$ en fonction de la mesure de sa surface ? de son volume ? Andrews [2] montre en 1963 l'existence d'une constante $c_d > 0$ telle que tout compact convexe K de $\mathbb{R}^d$ de volume $V > 0$ ayant n points non coplanaires à sa surface vérifie l'inégalité

$$n \leq c_d V^{\frac{d-1}{d+1}}.$$

Compte tenu de l'inégalité isopérimétrique, ceci entraîne immédiatement une majoration uniforme de n par $S^{d/(d+1)}$ où S désigne la mesure de surface de K.

1.2.1 La question d'Arnold

Deux polytopes convexes P et Q à sommets dans $\mathbb{Z}^d$ sont dits équivalents s'il existe un isomorphisme affine $T : \mathbb{Z}^d \to \mathbb{Z}^d$ tel que $T(P) = Q$. Puisqu'une telle transformation est nécessairement de déterminant ± 1, deux polytopes équivalents ont le même volume. On note $N_d(V)$ le nombre de classe d'équivalences de polytopes dont le volume est inférieur ou égal à V.

En lien avec l'étude des polygones de Newton des polynômes à plusieurs variables, Arnold [4] s'intéresse en 1980 au comportement asymptotique de $\log N_d(V)$ lorsque le volume V tend vers $+\infty$. Dans le cas du plan, il prouve l'existence de constantes $c_1, c_2 > 0$ telles que

$$c_1 V^{\frac{1}{3}} \le \log N_2(V) \le c_2 V^{\frac{1}{3}} \log V, \qquad V \to +\infty.$$

Sa démonstration repose sur un cas particulier de l'inégalité d'Andrews [2] citée précédemment, qu'Arnold retrouve indépendamment. Réciproquement, on peut redémontrer l'inégalité d'Andrews dans le cas $d = 2$ à partir de la minoration d'Arnold. Bárányet Pach [13] montrent que le facteur logarithmique est superflu.

Konyagin et Sevast'yanov [34] généralisent le résultat d'Arnold en prouvant pour tout $d \ge 2$ l'existence de constantes $c_1, c_2 > 0$ telle que

$$c_1 V^{\frac{d-1}{d+1}} \le \log N_d(V) \le c_2 V^{\frac{d-1}{d+1}} \log V, \qquad V \to +\infty.$$

Bárány et Vershik [9] montrent que le facteur logarithmique est à nouveau superflu dans le cas général. Les démonstrations de ces théorèmes font intervenir des idées issues de la théorie des nombres, de la géométrie convexe et de la géométrie des nombres.

Au vu de ces résultats, la question se pose de savoir s'il existe une constante $\alpha > 0$ telle que

$$\log N_d(V) \sim \alpha V^{\frac{d-1}{d+1}}, \qquad V \to +\infty.$$

La réponse n'est pas connue à ce jour, même dans le cas $d = 2$. Les spécialistes estiment que le problème est probablement difficile [10].

1.2.2 Forme limite des polygones convexes

Dans la démonstration d'Arnold, la majoration de $\log N_2(V)$ est obtenue par un lemme de *mise en boîte* : toutes les classes d'équivalences de polygones convexes à sommets entiers de volume au plus V ont au moins un représentant inclus dans le carré $[0;36V] \times [0;36V]$. Cette remarque nous amène à une version un peu différente du problème d'Arnold. On considère maintenant l'ensemble des polygones convexes à sommets entiers inclus dans le carré $[-n;n] \times [-n;n]$. Combien y en a-t-il et à quoi ressemblent-ils ?

La question a été posée par Vershik et la réponse a été donnée indépendamment par Vershik [53], Bárány [8] et Sinaï [44] pendant l'année 1994. Le logarithme du nombre de polygones convexes à sommets entiers inclus dans $[-n;n]^2$ est équivalent à

$$12\left(\frac{\zeta(3)}{\zeta(2)}\right)^{\frac{1}{3}} n^{\frac{2}{3}}.$$

Plus tard, Vershik et Zeitouni [56] ont démontré pour une classe de problèmes analogues un principe de grandes déviations dont la fonction de taux est donnée par le périmètre affine de la courbe.

Finalement, Acketa et Žunić [1] se sont intéressés au nombre maximal de sommets qu'un polygone convexe à sommets entiers inclus dans un carré de taille fixée peut avoir. Ils ont ainsi retrouvé un analogue du résultat de Jarník. Le problème plus général où le carré est remplacé par n'importe quel grand convexe compact a été traité par Bárány et Prodromou [14].

La nature singulière des résultats obtenus illustre le fait que cette classe de problèmes est reliée à la fois à la géométrie des nombres, à la géométrie différentielle affine et à la théorie des nombres. En effet, la parabole obtenue comme forme limite est l'unique solution du problème d'optimisation consistant à trouver la courbe convexe inscrite dans le carré ayant le plus grand périmètre affine. De plus, l'appartition de la fonction zêta de Riemann montre l'aspect arithmétique du problème. On pourrait en effet montrer que si on remplaçait le réseau $\mathbb{Z}^2$ par un processus ponctuel de Poisson d'intensité unité (auquel on peut penser comme au « réseau » le plus isotrope possible), alors avec grande probabilité, les constantes $(\zeta(3)^2\zeta(2))^{-1/3} \approx 0.749$ et $3(\zeta(3)/\zeta(2))^{1/3} \approx 2.702$ seraient simplement changées par 1 et 3.

1.2.3 Notre contribution

Nos travaux portent sur l'énumération asymptotique des lignes polygonales convexes à sommets entiers dans le plan. Nous établissons notamment un équivalent à la Hardy-Ramanujan pour ces objets. La formule obtenue présente des oscillations contrôlées par les zéros non triviaux de la fonction zêta, ce qui souligne le lien entre ce problème et l'hypothèse de Riemann. Nous présentons aussi une analyse combinatoire détaillée des lignes polygonales convexes tenant compte du nombre de sommets. Ceci nous permet de répondre à une question ouverte de Vershik concernant l'existence d'une forme limite pour les polygones convexes ayant peu de sommets. Enfin, nous introduisons une nouvelle famille de contraintes incluant le problème de Jarník et nous explicitons les formes limites associées.

Les recherches présentées dans cette section ont été réalisées en collaboration avec Nathanaël Enriquez.

Une formule à la Hardy-Ramanujan

Nous avons cherché à affiner l'estimation logarithmique du nombre de chaînes convexes obtenue par Vershik, Bárány et Sinaï. Notre stratégie repose sur l'approche probabiliste de Sinaï [44] qui utilise une description du problème en termes de modèle de mécanique statistique. La pierre angulaire de notre travail est une *représentation intégrale exacte* de la fonction de partition de ce modèle. Nous parvenons en effet à exprimer cette fonction sous la forme d'une intégrale de contour complexe qui fait intervenir la fonction zêta de Riemann :

$$\frac{1}{2i\pi}\int_{3-i\infty}^{3+i\infty}\frac{\Gamma(s)\zeta(s+1)(\zeta(s)+\zeta(s-1))}{\zeta(s)\beta^s}ds.$$

L'étude des résidus de cette intégrale et la théorie classique de la fonction zêta [50] nous donnent alors accès à des développements asymptotiques de la fonction de partition et des paramètres principaux du modèle. Enfin, nous démontrons à l'aide d'un théorème local limite l'analogue suivant de la formule de Hardy et Ramanujan pour les chaînes convexes.

Théorème. *Posons $\kappa = \zeta(3)/\zeta(2)$ et considérons la fonction S suivante, définie par sommation sur l'ensemble des zéros ρ de la fonction zêta de Riemann dans la bande critique $0 < \mathfrak{Re}(\rho) < 1$,*

$$S(x) = \sum_{\rho}\frac{\Gamma(\rho)\zeta(\rho+1)\zeta(\rho-1)}{\zeta'(\rho)}x^{\frac{\rho}{3}}, \qquad x > 0.$$

Le nombre de chaînes polygonales convexes à sommets dans $([0,n]\cap\mathbb{Z})^2$ reliant $(0,0)$ et (n,n) est asymptotiquement équivalent à

$$\frac{e^{-2\zeta'(-1)}}{(2\pi)^{7/6}\sqrt{3}\kappa^{1/18}}\frac{1}{n^{17/18}}e^{3\kappa^{1/3}n^{2/3}+S(\kappa/n)}.$$

Il est important de remarquer que, conditionnellement à l'hypothèse de Riemann selon laquelle tous les zéros de la fonction zêta dans la bande critique vérifient $\mathfrak{Re}(\rho) = \frac{1}{2}$, le terme oscillatoire $S(\kappa/n)$ est seulement d'ordre $n^{1/6}$. De façon inconditionnelle, nous savons seulement montrer que $S(\kappa/n) = o(n^{1/3})$.

Notons que l'écriture sommatoire de S suppose implicitement que la fonction zêta n'a pas de zéro multiple, ce qui reste une conjecture à ce jour. Cette hypothèse permet d'alléger les notations mais on pourrait s'en passer. En fait, on dispose d'une définition de S comme intégrale sur un contour complexe pour laquelle la question ne se pose pas. La convergence de l'écriture sommatoire est par ailleurs justifiée par un théorème de Valiron [50, Theorem 9.7] sur l'existence de segments horizontaux traversant la bande critique et sur lesquels la fonction ζ prend des valeurs suffisamment éloignées de 0.

Combinatoire détaillée

Dans le même esprit que Vershik et Yakubovich [55] qui dénombraient les partitions d'entiers ayant un nombre fixé de composantes, nous donnons un équivalent logarithmique du nombre de chaînes convexes ayant un nombre fixé de sommets. Le résultat que nous obtenons peut s'écrire :

Théorème. *Considérons les fonctions* $\mathbf{c}$ *et* $\mathbf{e}$ *définies par*

$$\mathbf{c}(\lambda) = \frac{1}{\zeta(2)^{1/3}} \frac{\lambda \operatorname{Li}_2(1-\lambda)}{(1-\lambda)(\zeta(3)-\operatorname{Li}_3(1-\lambda))^{2/3}}, \qquad \lambda > 0$$

et

$$\mathbf{e}(\lambda) = 3\left(\frac{\zeta(3)-\operatorname{Li}_3(1-\lambda)}{\zeta(2)}\right)^{1/3} - \mathbf{c}(\lambda)\log\lambda, \qquad \lambda > 0.$$

Soit $p(n;k)$ *le nombre de chaînes convexes à sommets dans* $([0,n]\cap\mathbb{Z})^2$ *reliant* $(0,0)$ *et* (n,n) *et ayant* k *sommets.*

— *S'il existe* $\lambda > 0$ *tel que* $k \sim \mathbf{c}(\lambda)n^{2/3}$*, alors*

$$\log p(n;k) \sim \mathbf{e}(\lambda)n^{2/3}.$$

— *Si* k *est négligeable devant* $n^{2/3}$*, alors*

$$p(n;k) = \left(\frac{n^2}{k^3}\right)^{k+o(k)}.$$

La courbe $(\mathbf{c}(\lambda),\mathbf{e}(\lambda))$ contient de nombreuses informations sur la distribution des chaînes convexes. Elle passe notamment par deux points remarquables :

— Le point $(\mathbf{c}(1),\mathbf{e}(1)) = ((\zeta(3)^2\zeta(2))^{-1/3}, 3(\zeta(3)/\zeta(2))^{1/3})$, où la coordonnée $\mathbf{e}(\lambda)$ est maximale, redonne le résultat de Vershik [53], Bárány [8] et Sinaï [44] sur le nombre total de chaînes convexes.

— Le point limite $(\mathbf{c}(\infty),\mathbf{e}(\infty)) = (3\pi^{-2/3}, 0)$, où la coordonnée $\mathbf{c}(\lambda)$ est maximale, redonne le résultat d'Acketa et Žunić [1] sur le nombre maximal de sommets d'une chaine convexe.

La preuve de ce théorème passe par l'introduction d'un nouveau paramètre $\lambda > 0$ dans le modèle probabiliste de Sinaï qui permet de tenir compte du nombre de sommets de la chaîne : conditionnellement à son extrémité, une chaîne convexe apparaîtra avec une probabilité proportionnelle à λ^k où k est son nombre de sommets. Il s'agit donc du modèle de Sinaï lorsque $\lambda = 1$, d'une pénalisation lorsque $0 < \lambda < 1$, et d'une récompense lorsque $\lambda > 1$.

Le cas d'un nombre k extrêmement petit de sommets, par exemple négligeable devant $\log n$, sort du domaine d'application de cette méthode « mécanique statistique ». Une approche élémentaire, analogue à celle d'Erdőset Lehner [23] pour les partitions d'entiers, permet heureusement d'obtenir un équivalent de $p(n;k)$ pour k négligeable devant $\sqrt{n}/(\log n)^{1/4}$.

Forme limite

L'étude de la forme limite des chaînes convexes aléatoires dans ces nouveaux modèles donne une réponse à une question ouverte posée par Vershik dans son article de 1994 concernant les chaînes ayant un nombre anormalement petit de sommets, c'est à dire négligeable devant $n^{2/3}$:

> « *Theorem 3.1 shows how the number of vertices of a typical polygonal line grows. However, one can consider some other fixed growth, say,* $\sqrt{n}$*, and look for the limit shapes for uniform distributions connected with this growth [...]* »

Il se trouve qu'il existe encore une forme limite dans ces régimes particuliers. Mieux, la forme limite est toujours le même arc de parabole. C'est le contenu du théorème suivant.

Théorème. *Soit* $\Gamma_{n,k}$ *une chaîne convexe aléatoire à sommets dans* $(\frac{1}{n}\mathbb{Z} \cap [0,1])^2$ *reliant* $(0,0)$ *et* $(1,1)$ *tirée uniformément parmi les chaînes ayant au plus k sommets. Si n et k tendent vers* $+\infty$*, alors* $\Gamma_{n,k}$ *converge en probabilité vers l'arc de parabole*

$$\{(x,y) \in [0,1]^2 : \sqrt{y} + \sqrt{1-x} = 1\}$$

pour la distance de Hausdorff.

L'essentiel de la preuve de ce résultat est basé sur l'approche « mécanique statistique » évoquée plus haut. Notons néanmoins que le cas d'un nombre k de sommets extrêmement petit, par exemple négligeable devant $\log n$, sort du domaine d'application de cette méthode. Pour traiter ce cas, nous proposons une approche plus élémentaire basée sur une comparaison avec un modèle continu très simple : choisir uniformément $k-1$ points dans $[0,1]^2$ conditionnés à être en position croissante et à ce que la ligne polygonale qu'ils définissent entre $(0,0)$ et $(1,1)$ soit convexe. Dans ce modèle continu, l'existence de la forme limite parabolique a été prouvée par Bárány [12] et Bárány, Rote, Steiger, Zhang [15].

1.3 Forme limite des partitions multi-partites et zonotopes

Les recherches présentées dans cette section font l'objet d'une collaboration en cours avec Imre Bárány.

On a déjà esquissé dans les sections précédentes quelques traces d'une étude des propriétés asymptotique des partitions entières multipartites et des grands polytopes convexes à sommets entiers dans l'espace affine $\mathbb{R}^d$ dans le cas de la dimension $d = 2$. Vershik [52] et Bárány [11] en présentent des aspects plus généraux.

Un résultat central de cette étude est celui de Vershik [53], Bárány [8] et Sinaĭ [44], qui ont démontré l'existence d'une forme limite asymptotique déterministe pour les polygones convexes à sommets entiers tirés uniformément à l'intérieur d'un grand carré $[-n;n]$, lorsque qu'on fait tendre n vers l'infini.

Leurs approches reposent toutes de manière essentielle sur une bijection entre l'ensemble des lignes polygonales convexes croissantes à sommets entiers d'une part, et l'ensemble des partitions *strictes* d'entiers bipartis d'autre part. Rappelons qu'une partition stricte de $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$ est une partition dans laquelle chacune des parts est un vecteur bidimensionnel dont les composantes sont des entiers naturels premiers entre eux. Ces couples d'entiers sont donnés par la liste des vecteurs décrivant les directions successives des côtés de la ligne polygonale.

Lorsqu'on passe en dimension $d \geq 3$, les polytopes convexes à sommets entiers à l'intérieur d'un hypercube $[-n, n]^d$ ne peuvent plus être représentés aussi simplement par une partition d'entiers d-partites. Mais il reste une correspondance partielle entre les partitions strictes et les zonotopes à sommets entiers, qui sont les polytopes convexes obtenu en considérant l'enveloppe convexe de l'ensemble des vecteurs

$$\sum_{i=1}^{k} \varepsilon_k \mathbf{v}_k, \qquad (\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k$$

pour une famille $\mathbf{v}_1, \ldots, \mathbf{v}_k$ de vecteurs de $\mathbb{Z}^d$ donnée.

Lorsque tous les vecteurs $\mathbf{v}_1, \ldots, \mathbf{v}_k$ sont situés du même côté d'un hyperplan vectoriel H, le vecteur $\mathbf{v}_1 + \cdots + \mathbf{v}_k$ est le point extrémal du zonotope qui se trouve à la distance maximale de l'hyperplan H. On l'appellera le *point final*. Même dans le cas, le nombre total de zonotopes avec un point final fixé est généralement infini. Mais si nous ajoutons la contrainte que les vecteurs doivent être pris dans un cône convexe saillant C, alors il n'y a qu'un nombre fini $p_C(\mathbf{n})$ de zonotopes ayant $\mathbf{n}$ pour point final. The cas du premier orthant $C = [0, \infty]^d$ est le plus naturel, mais notre approche reste pertinente pour un cône générique.

Nous nous sommes principalement intéressé au comportement asymptotique de la mesure uniforme sur l'ensemble des zonotopes à sommets entiers dont les vecteurs générateurs sont dans C et dont le point final est $\mathbf{n}$, lorsque $\|n\|$ tend vers l'infini. Le premier théorème que nous établissons donne un équivalent logarithmique du nombre $p_C(\mathbf{n})$. Le résultat principal de nos travaux réside en l'existence d'une forme limite asymptotique pour les zonotopes aléatoires tirés sous la loi uniforme. Aussi bien l'équivalent logarithmique que la forme limite sont décrits simplement à l'aide de la transformée de Laplace du cône C, donc nous introduisons la définition.

La démonstration de ces théorèmes repose sur une généralisation de la méthode probabiliste de Sinaĭ avec laquelle il a traité les chaînes convexe. Nous introduisons une mesure de probabilité sur l'ensemble des zonotopes à sommets entiers qui est analogue à l'ensemble grand canonique de la physique statistique à la Boltzmann. Plus précisément, la multiplicité $\omega(\mathbf{x})$ de chaque vecteur $\mathbf{x} \in C \cap \mathbb{Z}^d_*$ dans la partition décrivant le zonotope est tirée indépendamment selon une loi géométrique

$$\mathbb{P}_{\beta\mathbf{u}}(\omega(\mathbf{x}) = k) = (1 - e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle})e^{-k\beta\langle\mathbf{u},\mathbf{x}\rangle}, \qquad k \in \mathbb{N}$$

où $\beta > 0$ et $\mathbf{u} \in \mathbb{R}^d$ sont des paramètres du modèle qu'il reste à calibrer. La calibration naturelle consiste à les choisir de sorte que

$$\mathbb{E}_{\beta\mathbf{u}}\left(\sum_{\mathbf{x}\in C\cap\mathbb{Z}_*^d}\omega(\mathbf{x})\,\mathbf{x}\right)=\mathbf{n}+o(\|\mathbf{n}\|).$$

Le pont entre ce modèle et la mesure uniforme dépend d'un théorème central limite local générique, qui a été présenté puis appliqué aux partitions d'entiers bipartis dans [19].

Une autre direction d'étude concernant les formes limites de zonotopes a été considérée par Davydov and Vershik [21]. Celle-ci traite du cas où les vecteurs générateurs $\mathbf{v}_1,\dots,\mathbf{v}_k$ du zonotope sont des vecteurs aléatoires indépendants et de même loi. Ce modèle de zonotopes aléatoire est intimement lié aux réarrangments convexes de marches aléatoires. Davydov and Vershik interprètent ces zonotopes eux-même comme des marches aléatoires sur l'espace de Banach des sous-ensembles convexes de $\mathbb{R}^d$, où les sommations sont prises au sens de l'addition de Minkowski. Cette interprétation leur permet d'obtenir une loi des grands nombres un théorème central limite pour ces marches aléatoires.

Chapter 2

Partitions of large unbalanced bipartites

We compute the asymptotic behaviour of the number of partitions of large vectors (n_1, n_2) of $\mathbb{Z}_+^2$ in the critical regime $n_1 \asymp \sqrt{n_2}$ and in the subcritical regime $n_1 = o(\sqrt{n_2})$. This work completes the results established in the fifties by Auluck, Nanda, and Wright.

2.1 Introduction

How many ways are there to decompose a finite-dimensional vector whose components are non-negative integers as a sum of non-zero vectors of the same kind, up to permutation of the summands? The celebrated theory of integer partitions deals with the one-dimensional version of this problem. We refer the reader to the monograph of Andrews [3] for an account on the subject. In a famous paper, Hardy and Ramanujan [31] discovered the asymptotic behaviour of the number of partitions of a large integer n:

$$p_{\mathbb{N}}(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}. \tag{2.1}$$

In the two-dimensional setting, the number of partitions of a large vector has been studied by many authors, by various approaches. In the early fifties, the physicist Fermi [27] introduced thermodynamical models characterized by the conservation of two parameters instead of just one (corresponding to integer partitions). This led Auluck [6] to search for an asymptotic expression of the number of partitions of an integer vector (n_1, n_2). He established formulæ in two very different regimes. His first formula holds when n_1 is fixed and n_2 tends to infinity:

$$p_{\mathbb{Z}_+^2}(n_1, n_2) \sim \frac{1}{n_1!}\left(\frac{\sqrt{6n_2}}{\pi}\right)^{n_1} \frac{1}{4n_2\sqrt{3}} \exp\left\{\pi\sqrt{\frac{2n_2}{3}}\right\}. \tag{2.2}$$

His second one concerns the case where both components n_1 and n_2 tend to infinity with the same order of magnitude. The corresponding formula is much more involved but it can be simplified in the special case $n_1 = n_2 = n$. For some explicit constants a, b, c, one has

$$p_{\mathbb{Z}_+^2}(n,n) \sim \frac{a}{n^{\frac{55}{36}}} \exp\left\{b\, n^{\frac{2}{3}} + c\, n^{\frac{1}{3}}\right\}.$$

In the late fifties, Nanda [36] managed to extend the domain of validity of Auluck's first formula to the weaker condition $n_1 = o(n_2^{1/4})$. Shortly after, Wright [57] was able to prove that Auluck's second formula can be extended to the more general regime

$$\frac{1}{2} < \liminf \frac{\log n_1}{\log n_2} \leq \limsup \frac{\log n_1}{\log n_2} < 2.$$

Note finally that Robertson [41, 42] proved analogous formulae in higher dimensions.

Our article covers the case $n_1 = O(\sqrt{n_2})$, which completes these previous results. In particular, we deal with the case where n_1 and $\sqrt{n_2}$ have the same order of magnitude, which appears as a critical regime.

The papers of Auluck, Nanda and Wright all rely on generating function techniques. At the exception of Nanda's work, which is directly based on integer partition estimates, the main idea is to extract the asymptotic behaviour of the coefficients from the generating function with a Tauberian theorem or a saddle-point analysis. The extension proven by Wright was made possible by a more precise approximation of the generating function.

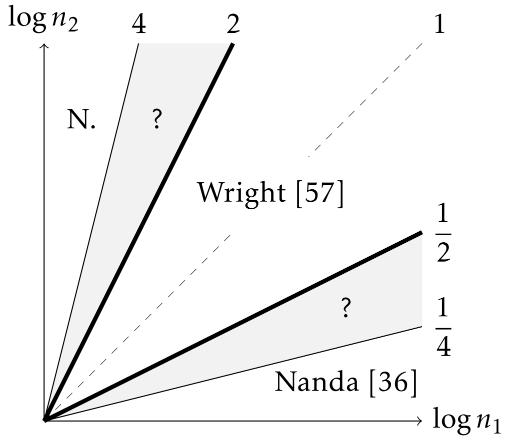

Figure 2.1 – Phase diagram of asymtptotic regimes. The previously studied asymptotic regions are indicated. Our work covers the unknown grey region, including the thick critical lines $\frac{1}{2}$ and 2, as well as Nanda's region.

The method we use in this article differs from the previous ones by relying heavily on a probabilistic embedding of the problem which is inspired by the Boltzmann model in statistical mechanics. The first ingredient of our proof is a precise estimate of the associated logarithmic partition function, based on an contour-integral representation of this function and Cauchy's residue theorem. The second ingredient is a bivariate local limit theorem, which follows from a general framework developed at the end of the paper.

Local limit theorems happen to play a crucial role in the treatment of questions from statistical mechanics, where they provide a rigorous justification of the *equivalence of ensembles* principle. Twenty years ago, Fristedt [28] introduced similar ideas to study the structure of uniformly drawn random partitions of large integers. A few years later, Báez-Duarte [7] applied a local limit theorem technique to derive a short proof of the Hardy-Ramanujan formula (2.1). The first implementation of these ideas in a two-dimensional context seems to be due to Sinaĭ [44], although the setting differs from ours. A general presentation of these techniques was discussed by Vershik [52]. In a recent paper, Bogachev and Zarbaliev [17] presented among other results a detailed proof of Sinaĭ's approach. Let us mention that the strong anisotropy which is inherent in the problem that we address makes the implementation of this program more delicate.

2.2 Notations and statement of the results

Let $\mathbb{Z}_+ = \{0,1,2,3,\ldots\}$ and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$ denote respectively the set of non-negative numbers and the set of positive integers.

We will use the standard Landau notations $a_n = o(b_n)$ or $a_n = O(b_n)$ for sequences (a_n) and (b_n) satisfying respectively $\limsup \left|\frac{a_n}{b_n}\right| = 0$ or $\limsup \left|\frac{a_n}{b_n}\right| < +\infty$. Also, we will write $a_n \asymp b_n$ if a_n and b_n have the same order of magnitude as n tends to infinity, that is to say if both $a_n = O(b_n)$ and $b_n = O(a_n)$ hold.

Definition. *Let X be a subset of $\mathbb{Z}_+^2$. For every $n \in \mathbb{Z}_+^2$, a* partition *of n with* parts *in X is a finite unordered family of elements of X whose sum is n. It can be represented by a* multiplicity *function $\omega\colon X \to \mathbb{Z}_+$ such that $\sum_{x\in X} \omega(x)\cdot x = n$. For $x \in X$, we say that $\omega(x)$ is the multiplicity of the part x in the partition. The partitions of n with parts in X constitute the set*

$$\Omega_X(n) := \left\{ \omega \in \mathbb{Z}_+^X : \sum_{x\in X} \omega(x)\cdot x = n \right\}.$$

Finally, we write $p_X(n) := |\Omega_X(n)|$ for the number of partitions of n with parts in X.

Following the works of Wright and Robertson [57, 41, 42], we will focus on two particular sets of parts in this article, namely:

- $X = \mathbb{N}^2$ which corresponds to partitions in which no part has a zero component,
- $X = \mathbb{Z}_+^2 \setminus \{0\}$ which corresponds to the case of general partitions, in which parts may have a zero component. We still have to exclude the zero part in order to ensure that every vector has only finitely many partitions.

The following theorem states the main results of the paper. It describes the asymptotic behaviour of $p_X(n)$ in the case of partitions without zero components as well as in the case of general partitions, outside Wright's region. First, we need to introduce the following auxiliary functions of $\alpha > 0$:

$$\Phi(\alpha) = \sum_{r\geq 1} \frac{1}{r^2} \frac{e^{-\alpha r}}{1-e^{-\alpha r}},\ \Theta(\alpha) = -\frac{\Phi'(\alpha)}{\sqrt{\Phi(\alpha)}},\ \overline{\Phi}(\alpha) = \Phi(\alpha) + \frac{\pi^2}{6},\ \overline{\Theta}(\alpha) = -\frac{\overline{\Phi}'(\alpha)}{\sqrt{\overline{\Phi}(\alpha)}},$$

$$\Psi(\alpha) = \sum_{r\geq 1} \frac{1}{r} \frac{e^{-\alpha r}}{1-e^{-\alpha r}},\ \Delta(\alpha) = 2\Phi(\alpha)\Phi''(\alpha) - \Phi'(\alpha)^2,\ \overline{\Delta}(\alpha) = 2\overline{\Phi}(\alpha)\overline{\Phi}''(\alpha) - \overline{\Phi}'(\alpha)^2.$$

Consider two sequences $(n_1(k))_k$ and $(n_2(k))_k$ of positive integers. In the sequel, the limits and asymptotic comparisons are to be understood as k approaches infinity. The index k will remain implicit.

Theorem 2.1. *Assume that both n_1 and n_2 tend to infinity under the conditions $n_1 = O(\sqrt{n_2})$ and $\log(n_2) = o(n_1)$.*

(i) If $\alpha_n > 0$ is the unique solution of $\Theta(\alpha_n) = \frac{n_1}{\sqrt{n_2}}$, then

$$p_{\mathbb{N}^2}(n_1, n_2) \sim \frac{1}{2\pi} \frac{\Phi(\alpha_n)}{n_2} \frac{e^{-\frac{1}{2}\Psi(\alpha_n)}}{\sqrt{\Delta(\alpha_n)}} \exp\left\{\left(\alpha_n \Theta(\alpha_n) + 2\sqrt{\Phi(\alpha_n)}\right)\sqrt{n_2}\right\}.$$

(ii) If $\alpha_n > 0$ is the unique solution of $\overline{\Theta}(\alpha_n) = \frac{n_1}{\sqrt{n_2}}$, then

$$p_{\mathbb{Z}_+^2\setminus\{0\}}(n_1, n_2) \sim \frac{1}{(2\pi)^{\frac{3}{2}}} \left(\frac{\overline{\Phi}(\alpha_n)}{n_2}\right)^{\frac{5}{4}} \frac{e^{\frac{1}{2}\Psi(\alpha_n)}}{\sqrt{\overline{\Delta}(\alpha_n)}} \exp\left\{\left(\alpha_n \overline{\Theta}(\alpha_n) + 2\sqrt{\overline{\Phi}(\alpha_n)}\right)\sqrt{n_2}\right\}.$$

We will present a complete proof of (i) and will state along the proof the additional arguments which are needed for (ii).

Although the formulæ in Theorem 2.1 involve an implicit function α_n of (n_1, n_2), remark that we can actually derive explicit expansions in terms of (n_1, n_2) when n_1 is negligible compared to $\sqrt{n_2}$, which condition is equivalent to $\alpha_n \to +\infty$. Notice indeed that the auxiliary functions $\Phi, \overline{\Phi}, \Theta, \overline{\Theta}, \Delta, \overline{\Delta}$, and Ψ admit simple asymptotic expansions in terms of the arithmetic function $\sigma_2(m) = \sum_{d|m} d^2$ as $\alpha \to +\infty$. This follows from the Lambert series [51, Section 4.71] elementary formulæ

$$\Phi(\alpha) = \sum_{m\geq 1} \frac{\sigma_2(m)}{m^2} e^{-\alpha m}, \qquad -\Phi'(\alpha) = \sum_{m\geq 1} \frac{\sigma_2(m)}{m} e^{-\alpha m}. \tag{2.3}$$

Asymptotic expansions of $\Theta^{-1}, \overline{\Theta}^{-1}$ can be computed effectively from there by an iterative method or by using the Lagrange reversion formula.

Let us show how these simples ideas allows us to extend the previous results by Nanda and Robertson. For example, the following application of case (i) provides additional corrective terms in the expansion given by Robertson [41, Theorem 2] which was stated for the special case $K = 1$, that is to say $n_1 = o(n_2^{1/3})$.

Corollary 2.1. *There exists a sequence (c_k) of rational numbers such that for all $K \in \mathbb{N}$, if n_1 and n_2 tend to $+\infty$ such that $n_1^{2K+1} = o(n_2^K)$ and $\log(n_2) = o(n_1)$,*

$$p_{\mathbb{N}^2}(n_1, n_2) \sim \frac{n_1}{n_2} \frac{n_2^{n_1}}{(n_1!)^2} \exp\left\{ \sum_{k=1}^{K-1} c_k \frac{n_1^{2k+1}}{n_2^k} \right\}.$$

Notice that the proof of this formula, which is given below, can be directly translated into an effective algorithm. For instance, we give here the first terms of the sequence (c_k) which have been computed with the help of the *Sage* mathematical software [45]:

$$c_1 = \frac{5}{4},\ c_2 = -\frac{805}{288},\ c_3 = \frac{6731}{576},\ c_4 = -\frac{133046081}{2073600},\ c_5 = \frac{170097821}{414720},\ \ldots$$

In the same way, an application of case (ii) of our theorem leads to an extension of formula (2.2) which was stated under the condition $n_1 = o(n_2^{1/4})$, or equivalently $K = 1$, in the work of Nanda [36].

Corollary 2.2. *There exists a sequence $(\overline{c}_k)$ of real numbers such that for all $K \in \mathbb{N}$, if n_1 and n_2 tend to $+\infty$ such that $n_1^{K+1} = o(n_2^{K/2})$ and $\log(n_2) = o(n_1)$,*

$$p_{\mathbb{Z}_+^2}(n_1, n_2) \sim \frac{1}{n_1!} \left(\frac{\sqrt{6n_2}}{\pi} \right)^{n_1} \frac{1}{4n_2\sqrt{3}} \exp\left\{ \pi\sqrt{\frac{2n_2}{3}} \right\} \exp\left\{ \sum_{k=1}^{K-1} \overline{c}_k \frac{n_1^{k+1}}{n_2^{k/2}} \right\}$$

An effective computation of the first terms of the sequence $(\overline{c}_k)$ gives, with $a = \sqrt{\zeta(2)}$,

$$\overline{c}_1 = \frac{5a}{4} - \frac{1}{4a},\ \overline{c}_2 = \frac{5}{8} - \frac{145a^2}{72},\ \overline{c}_3 = 6a^3 - \frac{1385a}{576} + \frac{5}{32a} + \frac{1}{192a^3},\ \ldots$$

Proof of Corollary 2.1. As noted above, the condition $n_1 = o(\sqrt{n_2})$ implies that α_n tends to $+\infty$ (see the proof of Proposition 2.5 for details). Since the both of $\Phi(\alpha)$ and $\sqrt{\Delta(\alpha)}$ are equivalent to $e^{-\alpha}$ as α tends to $+\infty$, and $\Psi(\alpha)$ tends to 0, the non-exponential factor in the formula for $p_{\mathbb{N}^2}(n_1, n_2)$ of Theorem 2.1 reduces to $\frac{1}{2\pi n_2}$. Also, the identities (2.3) for $\Phi(\alpha)$ and $-\Phi'(\alpha)$ show that we can now work with the formal power series

$$f(z) = \sum_{m=1}^{\infty} \frac{\sigma_2(m)}{m^2} z^m, \quad g(z) = \frac{1}{f(z)} \left(-z \frac{d}{dz} f(z) \right)^2.$$

Namely, the equation $\Theta(\alpha_n) = n_1/\sqrt{n_2}$ corresponds formally to $g(z_n) = n_1^2/n_2$ with $z_n := e^{-\alpha_n}$. Since $g(z)$ has no constant term, we obtain by reversion an infinite asymptotic expansion (we use here the Poincaré notation $\sim$ to denote an asymptotic series):

$$e^{-\alpha_n} \sim \frac{n_1^2}{n_2} + \sum_{k=2}^{\infty} a_k \frac{n_1^{2k}}{n_2^k}.$$

for some sequence of rationnals (a_k) expressible with the help of the Lagrange inversion formula. From this point, it is now easy to derive the existence of two expansions (where (b_k) and (b'_k) are two sequences of rational numbers)

$$\alpha_n \sim \log n_2 - 2\log n_1 + \sum_{k=1}^{\infty} b_k \frac{n_1^{2k}}{n_2^k}, \quad \sqrt{\Phi(\alpha_n)} \sim \frac{n_1}{\sqrt{n_2}}\left[1 + \sum_{k=2}^{\infty} b'_k \frac{n_1^{2k}}{n_2^k}\right],$$

which, together with Theorem 2.1 and Stirling's formula, prove the result. □

In both Corollary 2.1 and Corollary 2.2, notice that the boundary of the domain of validity for the expansions is actually asymptotic, as K grows larger, to the critical regime $n_1 \asymp \sqrt{n_2}$ which is represented by the thick line in Figure 2.1 on page 24. In this critical case, Theorem 2.1 still applies but does not lead to any much simpler expression since, when $n_1/\sqrt{n_2}$ converges quickly enough to some positive constant, all terms depending on α_n tend to constant coefficients. Still, the theorem provides the existence and some expressions of the exponential rate functions

$$h(t) := \lim_{n\to+\infty} \frac{1}{\sqrt{n}} \log p_{\mathbb{N}^2}(\lfloor t\sqrt{n}\rfloor, n), \quad \overline{h}(t) = \lim_{n\to+\infty} \frac{1}{\sqrt{n}} \log p_{\mathbb{Z}_+^2\setminus\{0\}}(\lfloor t\sqrt{n}\rfloor, n),$$

which are defined for all $t > 0$. Figure 2.2 on the next page shows the graphs of these functions.

2.3 The probabilistic model

In this section, we introduce a family of Gibbs probability measures on the set $\Omega_X := \bigcup_{n\in\mathbb{Z}_+^2} \Omega_X(n)$ of all partitions, given some fixed set of parts X. The idea is that, while the uniform distribution on the set $\Omega_X(n)$ of partitions on n is hard to describe, it is much easier to define a distribution on the larger space Ω_X, that will give the exact same weight to every partition of n. Let $\alpha, \beta \in (0, +\infty)$ be two *shape parameters* to be chosen later and write $\lambda = (\alpha, \beta)$. To each choice of λ, we associate a probability measure $\mathbb{P}_\lambda$ on the discrete space Ω_X such that for every $\omega \in \Omega_X$,

$$\mathbb{P}_\lambda(\omega) := \frac{\exp\{-\sum_{x\in X} \omega(x)\langle\lambda, x\rangle\}}{\sum_{\omega\in\Omega_X} \exp\{-\sum_{x\in X} \omega(x)\langle\lambda, x\rangle\}}.$$

For each partition $\omega \in \Omega_X$, let us introduce the key quantity $N(\omega) := \sum_{x\in X} \omega(x)\cdot x$, which we will see as a random variable with values in $\mathbb{Z}_+^2$. By definition, we have $\omega \in \Omega_X(N(\omega))$

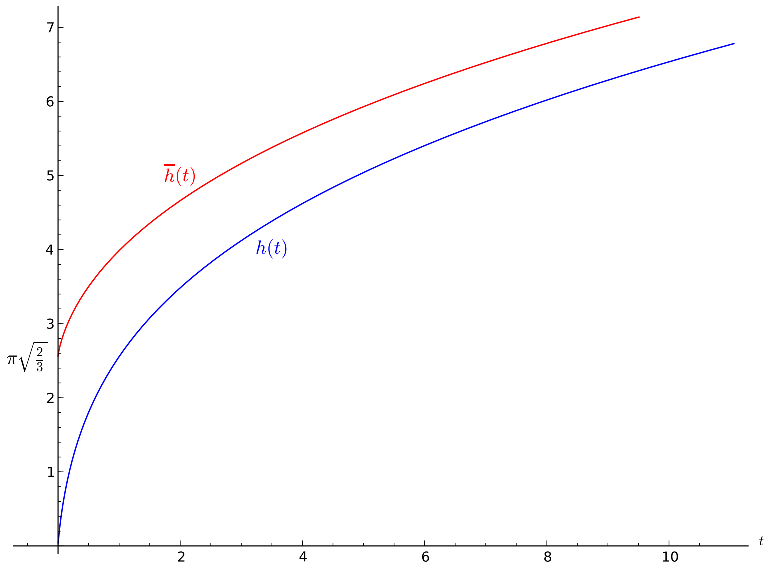

Figure 2.2 – Comparison of the rate functions h and $\overline{h}$ corresponding respectively to partitions without zero components and to partitions where zero components are allowed.

for every partition ω. Furthermore, the probability $\mathbb{P}_\lambda(\omega)$ now writes

$$\mathbb{P}_\lambda(\omega) = \frac{1}{Z_\lambda} e^{-\langle \lambda, N(\omega)\rangle}, \qquad \text{where } Z_\lambda := \sum_{\omega \in \Omega_X} e^{-\langle \lambda, N(\omega)\rangle}.$$

The normalization constant Z_λ is usually referred to as the *partition function* of the system in the statistical mechanics literature.

In accordance with the previous discussion, remark that the conditional distribution of $\mathbb{P}_\lambda$ on $\Omega_X(n)$ yields the uniform measure since the probability of a partition ω only depends on $N(\omega)$. Moreover, since all partitions of $\Omega_X(n)$ are given equal weight $\frac{1}{Z_\lambda} e^{-\langle \lambda, n\rangle}$, the total weight of $\Omega_X(n)$ is equal to $\frac{1}{Z_\lambda} p_X(n) e^{-\langle \lambda, n\rangle}$ and therefore

$$p_X(n) = \frac{Z_\lambda}{e^{-\langle \lambda, n\rangle}} \mathbb{P}_\lambda(N = n). \tag{2.4}$$

Probabilistic intuition dictates our strategy: calibrate the parameter λ as a function of n so that the distribution of the random vector N concentrates around n under the probability measure $\mathbb{P}_\lambda$. This way, we will be able to ensure a polynomial decrease for the quantity $\mathbb{P}_\lambda(N = n)$. A natural choice to enforce this behaviour is to take $\lambda_n = (\alpha_n, \beta_n)$ such that $\mathbb{E}_\lambda(N)$ is close enough to n. This is achieved by choosing the couple (α_n, β_n) defined by the equations (2.7) of Section 2.5. Looking back to $p_X(n)$, we see that we need to estimate precisely the partition function Z_λ as well as $\mathbb{P}_\lambda(N = n)$. The former will be done by a careful approximation of $\log Z_\lambda$ in section 4 and the latter will be deduced from estimates of the first and second derivatives of $\log Z_\lambda$ together with a Gaussian local limit theorem statement proven in section 5.

Before we turn to more technical discussions, let us remark that under the probability measure $\mathbb{P}_\lambda$, the random variables $\omega(x)$ for $x \in X$ are mutually independent and that their distribution is geometric. More precisely, we have for all $k \in \mathbb{Z}_+$,

$$\mathbb{P}_\lambda(\omega(x) = k) = e^{-k\langle \lambda, x\rangle}(1 - e^{-\langle \lambda, x\rangle}).$$

Finally, a fruitful consequence of the independence in this model is the fact that the partition function Z_λ can be written as an infinite product:

$$Z_\lambda = \prod_{x \in X} \frac{1}{1 - e^{-\langle \lambda, x\rangle}}. \tag{2.5}$$

Let us mention that (2.5) is the bipartite partition analogue of the famous Euler product formula for the usual partition generating function.

2.4 Approximation of the logarithmic partition function

Because of the product formula (2.5), the logarithm of the partition function Z_λ can be expressed as the sum of an absolutely convergent series:

$$\log Z_\lambda = -\sum_{x \in X} \log(1 - e^{-\langle \lambda, x\rangle}) = \sum_{x \in X} \sum_{r=1}^{\infty} \frac{e^{-r\langle \lambda, x\rangle}}{r}.$$

Let us recall that we consider the case $X = \mathbb{N}^2$ of partitions whose parts have non-zero components. The logarithmic partition function thus writes

$$\log Z_\lambda = \sum_{x_1 \geq 1} \sum_{x_2 \geq 1} \sum_{r \geq 1} \frac{e^{-\alpha x_1 r}}{r} e^{-\beta x_2 r}. \tag{2.6}$$

Let ζ and Γ denote respectively the Riemann zeta function and the Euler gamma function [51]. Also consider for every $\alpha > 0$ and $s \in \mathbb{C}$ the Dirichlet series $D_\alpha(s)$ defined by

$$D_\alpha(s) := \sum_{k \geq 1} \sum_{r \geq 1} \frac{e^{-\alpha k r}}{r^s} = \sum_{r \geq 1} \frac{1}{r^s} \frac{e^{-\alpha r}}{1 - e^{-\alpha r}}.$$

Recalling that the Cahen-Mellin inversion formula [51] yields for every $c > 0$ and $t > 0$,

$$e^{-t} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) t^{-s} \, ds,$$

we can rewrite the identity (2.6) for every $c > 1$ as

$$\begin{aligned}\log Z_\lambda &= \frac{1}{2i\pi} \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \sum_{r=1}^{\infty} \int_{c-i\infty}^{c+i\infty} \frac{e^{-r\alpha x_1}}{r} \Gamma(s)(r\beta x_2)^{-s}\, ds \\ &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)D_\alpha(s+1)\frac{ds}{\beta^s},\end{aligned}$$

the exchange in the order of summation being justified by the Fubini theorem. We proved that the logarithmic partition function admits an integral representation of the form

$$\log Z_\lambda = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} M_\alpha(s)\frac{ds}{\beta^s},$$

where $M_\alpha(s) := \Gamma(s)\zeta(s)D_\alpha(s+1)$. We will now see how to recover from the residues of the meromorphic function M_α the asymptotic behaviour of $\log Z_\lambda$ and its derivatives when β tends towards 0.

Proposition 2.3. *For every non-negative integers m, p, q, there exists a decreasing function $C_m^{p,q}(\alpha)$ of $\alpha > 0$ with a positive limit as $\alpha \to \infty$, such that the remainder function $R_m(\alpha,\beta)$ defined by*

$$R_m(\alpha,\beta) := \log Z_{(\alpha,\beta)} - \frac{D_\alpha(2)}{\beta} - \sum_{k=0}^{m} \frac{(-1)^k \zeta(-k) D_\alpha(1-k)}{k!}\beta^k$$

satisfies $|\partial_\alpha^p \partial_\beta^q R_m(\alpha,\beta)| \le C_m^{p,q}(\alpha)\, e^{-\alpha} \beta^{m-q+\frac{1}{2}}$ for all $\alpha, \beta > 0$.

Proof. Let us recall that the Riemann function $\zeta(s)$ is meromorphic on $\mathbb{C}$ and that it has a unique pole at $s = 1$, at which the residue is 1. The Euler gamma function $\Gamma(s)$ is also meromorphic on $\mathbb{C}$ and has poles at every integer $k \le 0$.

Let m be a positive integer and $\gamma_m = -m - \frac{1}{2}$. We are going to apply Cauchy's residue theorem to the meromorphic function M_α with the rectangular contour C defined by the segments $[2 - iT, 2 + iT]$, $[2 + iT, \gamma_m + iT]$, $[\gamma_m + iT, \gamma_m - iT]$ and $[\gamma_m - iT, 2 - iT]$, where $T > 0$ is some positive real we will let go to infinity. Computation of the residues of M_α in this stripe yields

$$\frac{1}{2i\pi} \int_{C^+} M_\alpha(s)\frac{ds}{\beta^s} = \frac{D_\alpha(2)}{\beta} + \sum_{k=0}^{m} \frac{(-1)^k \zeta(-k) D_\alpha(1-k)}{k!}\beta^k.$$

In order to prove that the contributions of the horizontal segments in the left-hand side integral vanish for $T \to \infty$, we use the three following facts:

(i) From the complex version of Stirling's formula, we know that $|\Gamma(\sigma + i\tau)|$ decreases exponentially fast when $|\tau|$ tends to $+\infty$, uniformly in every bounded stripe [51, p. 151].

(ii) Also $|\zeta(\sigma + i\tau)|$ is polynomially bounded in $|\tau|$ as $|\tau| \to +\infty$, uniformly in every bounded stripe [50, p. 95].

(iii) Finally, note that for all $\sigma_0 \in \mathbb{R}$,

$$\sup_{\sigma \geq \sigma_0} |D_\alpha(\sigma + i\tau)| \leq D_\alpha(\sigma_0) = \sum_{r \geq 1} \frac{1}{r^{\sigma_0}} \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} < \infty.$$

From these observations, we see that the integrals along horizontal lines vanish as $T \to \infty$:

$$\lim_{T \to +\infty} \int_{2+iT}^{\gamma_m + iT} M_\alpha(s) \frac{ds}{\beta^s} = \lim_{T \to +\infty} \int_{\gamma_m - iT}^{2-iT} M_\alpha(s) \frac{ds}{\beta^s} = 0.$$

Furthermore, $M_\alpha(s)\beta^{-s}$ is integrable on the vertical line $(\gamma_m - i\infty, \gamma_m + i\infty)$, so that in the limit $T \to \infty$, we obtain

$$\log Z_\lambda = \frac{D_\alpha(2)}{\beta} + \sum_{k=0}^{m} \frac{(-1)^k \zeta(-k) D_\alpha(1-k)}{k!} \beta^k + \frac{1}{2i\pi} \int_{\gamma_m - i\infty}^{\gamma_m + i\infty} M_\alpha(s) \frac{ds}{\beta^s}.$$

Thus, the remainder function $R_m(\alpha, \beta)$ is actually equal to the integral term in the right-hand side. We need to control its derivatives. Note that the derivatives

$$\partial_\alpha^p \partial_\beta^q \frac{M_\alpha(s)}{\beta^s} = (-1)^q \frac{\Gamma(s+q)}{\beta^{s+q}} \zeta(s) \partial_\alpha^p D_\alpha(s+1)$$

are well defined and integrable on the vertical line $(\gamma_m - i\infty, \gamma_m + i\infty)$ thanks again to the facts (i) and (ii), as well as the analogue of (iii) for the ∂_α^p derivative of D_α. It is easy to check that the result follows with

$$C_m^{p,q}(\alpha) e^{-\alpha} = \frac{|\partial_\alpha^p D_\alpha(\gamma_m + 1)|}{2\pi} \int_{-\infty}^{\infty} |\zeta(\gamma_m + i\tau)| |\Gamma(\gamma_m + q + i\tau)| \, d\tau. \qquad \square$$

Remark. *In the case $X = \mathbb{Z}_+^2 \setminus \{0\}$, the logarithmic partition function becomes* $\log Z_\lambda + \Psi(\alpha) + \Psi(\beta)$, *where $\Psi(\cdot) = D_\cdot(1)$ is the function defined in Section 2.2. The two additional terms correspond respectively to horizontal and vertical one-dimensional integer partitions. In that case, the expansion as $\beta \to 0^+$ involves the expansion of $\Psi(\beta)$ (which was the basis of [31]) and can be written informally as*

$$\frac{\Phi(\alpha) + \zeta(2)}{\beta} + \frac{1}{2} \log \beta + \frac{\Psi(\alpha)}{2} - \frac{1}{2} \log(2\pi) + \left(\frac{D_\alpha(0)}{12} - \frac{1}{24} \right) \beta + \ldots$$

2.5 Calibration of the shape parameters

In this section, we find appropriate values for the parameters $\lambda = (\alpha, \beta)$ as functions of n for which $\mathbb{E}_\lambda(N)$ is asymptotically close to n. Since the distribution of the random vector N under $\mathbb{P}_\lambda$ is given by a Gibbs measure,

$$\mathbb{E}_\lambda(N) = -\mathrm{grad}\,(\log Z_\lambda) = -\begin{bmatrix}\partial_\alpha \log Z_\lambda \\ \partial_\beta \log Z_\lambda\end{bmatrix}.$$

Let us recall that by definition, the function Φ introduced in Section 2.2 is

$$\forall \alpha > 0, \qquad \Phi(\alpha) := \sum_{r\geq 1} \frac{1}{r^2}\frac{e^{-\alpha r}}{1-e^{-\alpha r}},$$

which is exactly $D_\alpha(2)$ for the Dirichlet series $D_\alpha(s)$ introduced in Section 2.4. The approximation given in Proposition 2.3 applied with $(m,p,q) = (1,1,0)$ and $(m,p,q) = (1,0,1)$ yields the existence of two decreasing functions C_1 and C_2 with finite limits as $\alpha \to +\infty$ such that

$$\left|\partial_\alpha \log Z_\lambda - \frac{\Phi'(\alpha)}{\beta}\right| \leq e^{-\alpha} C_1(\alpha) \qquad \text{and} \qquad \left|\partial_\beta \log Z_\lambda + \frac{\Phi(\alpha)}{\beta^2}\right| \leq e^{-\alpha} C_2(\alpha).$$

This justifies the choice made in the next proposition to define the shape parameters α_n and β_n through the implicit equations (2.7). We start with a lemma.

Lemma 2.4. *The function Φ is logarithmically convex on* $(0;+\infty)$.

Proof. The function $\log\Phi$ being smooth, its convexity is equivalent to the inequality

$$\frac{d^2}{d\alpha^2}(\log\Phi(\alpha)) = \frac{\Phi(\alpha)\Phi''(\alpha) - \Phi'(\alpha)^2}{\Phi(\alpha)^2} \geq 0,$$

which follows immediately from the Cauchy-Schwarz inequality since

$$\Phi(\alpha) = \sum_{k\geq 1}\sum_{r\geq 1} \frac{e^{-\alpha k r}}{r^2}, \qquad \Phi'(\alpha) = -\sum_{k\geq 1}\sum_{r\geq 1} \frac{k e^{-\alpha k r}}{r} \quad \text{and} \quad \Phi''(\alpha) = \sum_{k\geq 1}\sum_{r\geq 1} k^2 e^{-\alpha k r}. \qquad \square$$

Proposition 2.5. *For all $n = (n_1, n_2) \in \mathbb{N}^2$, there exists a unique couple $(\alpha_n, \beta_n) \in (0;+\infty)^2$ such that*

$$\frac{-\Phi'(\alpha_n)}{\beta_n} = n_1 \qquad \textit{and} \qquad \frac{\Phi(\alpha_n)}{\beta_n^2} = n_2. \tag{2.7}$$

Proof. Eliminating β_n in the implicit equations (2.7), we need only prove the existence of a unique $\alpha_n > 0$ such that

$$\frac{1}{2}\frac{\Phi'(\alpha_n)}{\sqrt{\Phi(\alpha_n)}} = -\frac{1}{2}\frac{n_1}{\sqrt{n_2}}.$$

We recognize the derivative of the function $\sqrt{\Phi(\alpha)} = \exp\left(\frac{1}{2}\log\Phi(\alpha)\right)$ which is strictly convex by Lemma 2.4, so that $\Phi'/\sqrt{\Phi}$ is continuously increasing. In addition, it is easy to check that

$$\lim_{\alpha\to 0^+}\frac{\Phi'(\alpha)}{\sqrt{\Phi(\alpha)}}=\lim_{\alpha\to 0^+}\frac{-\zeta(3)/\alpha^2}{\sqrt{\zeta(3)/\alpha}}=-\infty \quad\text{and}\quad \lim_{\alpha\to+\infty}\frac{\Phi'(\alpha)}{\sqrt{\Phi(\alpha)}}=\lim_{\alpha\to+\infty}\frac{-e^{-\alpha}}{\sqrt{e^{-\alpha}}}=0.$$

This concludes the proof. □

Recall that the assumptions of Theorem 2.1 are $n_1 \to +\infty$ and $n_2 \to +\infty$ with the conditions $n_1 = O(\sqrt{n_2})$ and $\log(n_2) = o(n_1)$. Consider the couple of parameters α_n and β_n defined by the equations (2.7). A consequence of the proof above is that, under these assumptions, the sequence α_n is bounded away from 0 (but not from $+\infty$). In addition,

$$\frac{e^{-\alpha_n}}{\beta_n} \asymp n_1, \qquad \frac{e^{-\alpha_n}}{\beta_n^2} \asymp n_2, \qquad \text{and} \qquad \beta_n \asymp \frac{n_1}{n_2}. \tag{2.8}$$

2.6 The local limit theorem and its application

In the sequel, the parameters $\lambda_n = (\alpha_n, \beta_n)$ are chosen according to the equations (2.7). The aim of the present section is to show that the random vector N satisfies a local limit theorem. Sufficient conditions for such a theorem to hold are given in Proposition 2.11 of the next section. We check that these conditions are satisfied for our model. Finally, an application of Proposition 2.11 leads to the proof of Theorem 2.1.

2.6.1 An estimate of the covariance matrix

The assumptions of Proposition 2.11 require a good estimate of the covariance matrix Γ_{λ_n} of the random vector N under the measure $\mathbb{P}_{\lambda_n}$. Since we have a Boltzmann-type model with a Gibbs measure $\mathbb{P}_\lambda$, the covariance matrix Γ_λ of N is simply given by the second derivatives of the logarithmic partition function,

$$\Gamma_\lambda = \operatorname{Hess}(\log Z_\lambda) = \mathbb{E}_\lambda\left(\begin{bmatrix} \partial_\alpha^2 \log Z_\lambda & \partial_\alpha\partial_\beta \log Z_\lambda \\ \partial_\alpha\partial_\beta \log Z_\lambda & \partial_\beta^2 \log Z_\lambda \end{bmatrix}\right).$$

Denoting by Σ_λ the symmetric matrix

$$\Sigma_\lambda := \begin{bmatrix} \dfrac{\Phi''(\alpha)}{\beta} & -\dfrac{\Phi'(\alpha)}{\beta^2} \\ -\dfrac{\Phi'(\alpha)}{\beta^2} & \dfrac{2\Phi(\alpha)}{\beta^3} \end{bmatrix},$$

an application of Proposition 2.3 with $m = 2$ and $p + q = 2$ yields the existence of a positive decreasing function $C(\alpha)$ with a positive limit as $\alpha \to \infty$ such that

$$\|\Gamma_\lambda - \Sigma_\lambda\| \le e^{-\alpha} C(\alpha). \tag{2.9}$$

We can now state two crucial consequences of this approximation concerning Γ_{λ_n}. The first one concerns the precise asymptotic behaviour of its determinant while the second one shows that its eigenvalues go to $+\infty$. Let us recall that the function $\Delta(\alpha)$ introduced in Section 2.2 is defined by

$$\Delta(\alpha) := 2\Phi(\alpha)\Phi''(\alpha) - \Phi'(\alpha)^2 = \det\begin{bmatrix} \Phi''(\alpha) & -\Phi'(\alpha) \\ -\Phi'(\alpha) & 2\Phi(\alpha) \end{bmatrix}.$$

Proposition 2.6. *Under the assumptions of Theorem 2.1,* $\det\Gamma_{\lambda_n} \sim \dfrac{\Delta(\alpha_n)}{\beta_n^4} \asymp (n_2)^2$.

Proof. Using the approximation (2.9) and the fact that for all integers $p \geq 0$, one has $\Phi^{(p)}(\alpha) = (-1)^p e^{-\alpha} + O(e^{-2\alpha})$ as $\alpha \to +\infty$, we need to prove that $\Delta(\alpha)$ does not vanish for $\alpha > 0$ and that $e^{-2\alpha_n}/\beta_n^3$ is negligible compared to $\Delta(\alpha_n)/\beta_n^4$. By Lemma 2.4, Φ is logarithmically convex, so that we have $\Phi'(\alpha)^2 \leq \Phi''(\alpha)\Phi(\alpha)$. As a consequence, $\Delta(\alpha) > 0$ for all $\alpha > 0$. In addition, the estimates (2.8) imply that

$$\frac{\Delta(\alpha_n)}{\beta_n^4} \asymp \frac{e^{-2\alpha_n}}{\beta_n^4} \asymp (n_2)^2 \qquad \text{and} \qquad \frac{e^{-2\alpha_n}}{\beta_n^3} \asymp n_1 n_2,$$

which is enough to conclude because $n_1 = O(\sqrt{n_2})$ is negligible compared to n_2. □

Let us denote by $\Gamma_\lambda(x,x) = \langle x, \Gamma_\lambda x\rangle$ for $x \in \mathbb{R}^2$ the quadratic form on $\mathbb{R}^2$ induced by the symmetric positive-definite matrix Γ_λ.

Proposition 2.7. *Under the assumptions Theorem 2.1, there exist positive constants C_1, C_{-1} such that for all $x \in \mathbb{R}^2$,*

$$\Gamma_{\lambda_n}(x,x) \leq C_1\left(n_1|x_1|^2 + \frac{(n_2)^2}{n_1}|x_2|^2\right), \tag{2.10}$$

$$\Gamma_{\lambda_n}^{-1}(x,x) \leq C_{-1}\left(\frac{1}{n_1}|x_1|^2 + \frac{n_1}{(n_2)^2}|x_2|^2\right). \tag{2.11}$$

Proof. Let us first prove that the inequality (2.10) holds for Σ_{λ_n} instead of Γ_{λ_n}. For every $x = (x_1, x_2) \in \mathbb{R}^2$, the log-convexity of Φ (Lemma 2.4) $|\Phi'(\alpha)|^2 \leq \Phi''(\alpha)\Phi(\alpha)$ and the inequality between arithmetic mean and geometric mean yields

$$\left|\frac{\Phi'(\alpha)}{\beta^2}x_1x_2\right| \leq \sqrt{\frac{\Phi''(\alpha)}{\sqrt{2}\beta}|x_1|^2}\sqrt{\frac{\sqrt{2}\Phi(\alpha)}{\beta^3}|x_2|^2} \leq \frac{1}{2\sqrt{2}}\frac{\Phi''(\alpha)}{\beta}|x_1|^2 + \frac{\sqrt{2}}{2}\frac{\Phi(\alpha)}{\beta^3}|x_2|^2,$$

which implies that for the positive constants $C_\pm = 1 \pm \frac{1}{\sqrt{2}}$ and for all $x \in \mathbb{R}^2$,

$$C_-\left[\frac{\Phi''(\alpha)}{\beta}|x_1|^2 + \frac{2\Phi(\alpha)}{\beta^3}|x_2|^2\right] \leq \Sigma_\lambda(x,x) \leq C_+\left[\frac{\Phi''(\alpha)}{\beta}|x_1|^2 + \frac{2\Phi(\alpha)}{\beta^3}|x_2|^2\right].$$

In other words, the following matrix inequality holds for the Löwner ordering $\preceq$ (let us recall that two symmetric real matrices A and B satisfy $A \preceq B$ if $B - A$ is positive semi-definite):

$$C_- \begin{bmatrix} \frac{\Phi''(\alpha)}{\beta} & 0 \\ 0 & \frac{2\Phi(\alpha)}{\beta^3} \end{bmatrix} \preceq \Sigma_\lambda \preceq C_+ \begin{bmatrix} \frac{\Phi''(\alpha)}{\beta} & 0 \\ 0 & \frac{2\Phi(\alpha)}{\beta^3} \end{bmatrix}.$$

Considering the right-hand side of this inequality, and remembering that as a consequence of (2.8), we have $\Phi''(\alpha_n)/\beta_n \asymp n_1$ and $\Phi(\alpha_n)/\beta_n^3 \asymp (n_2)^2/n_1$, we see that the analogue of the bound (2.10) for the quadratic form Σ_{λ_n} holds. In order to complete the proof of the proposition, we need to control the error made when we replace Γ_{λ_n} by Σ_{λ_n}. Using (2.9), for all $x \in \mathbb{R}^2$,

$$\left|\Gamma_{\lambda_n}(x,x) - \Sigma_{\lambda_n}(x,x)\right| \le \|\Gamma_{\lambda_n} - \Sigma_{\lambda_n}\| \cdot \|x\|^2 \le e^{-\alpha_n} C(\alpha_n)\|x\|^2.$$

Under the conditions of Theorem 2.1, since $C(\alpha_n) \asymp 1$ and $e^{-\alpha_n} \asymp (n_1)^2/n_2$ which is negligible compared to both n_1 and $(n_2)^2/n_1$, we obtain

$$\frac{C_-}{2} \begin{bmatrix} n_1 & 0 \\ 0 & \frac{(n_2)^2}{n_1} \end{bmatrix} \preceq \Gamma_{\lambda_n} \preceq 2C_+ \begin{bmatrix} n_1 & 0 \\ 0 & \frac{(n_2)^2}{n_1} \end{bmatrix}, \tag{2.12}$$

which in turn implies, using the decreasing property of matrix inversion with respect to Löwner ordering,

$$\frac{1}{2C_+} \begin{bmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{n_1}{(n_2)^2} \end{bmatrix} \preceq \Gamma_{\lambda_n}^{-1} \preceq \frac{2}{C_-} \begin{bmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{n_1}{(n_2)^2} \end{bmatrix}. \tag{2.13}$$

The right-hand sides of (2.12) and (2.13) provide respectively (2.10) and (2.11). □

Corollary 2.8. *Let σ_n^2 be the smallest eigenvalue of Γ_{λ_n}. It satisfies $\sigma_n^2 \asymp n_1$.*

Proof. This is an immediate consequence of the inequalities (2.12). □

2.6.2 The condition on the Lyapunov ratio

We now check the second assumption of Proposition 2.11 below. Let $\Gamma_\lambda^{1/2}$ be the uniquely defined symmetric positive-definite square root of Γ_λ. We introduce the following analogue of the scale-independent Lyapunov ratio [16, p. 59]:

$$L_\lambda := \sup_{t \in \mathbb{R}^d} \frac{1}{\|\Gamma_\lambda^{1/2} t\|^3} \sum_{x \in X} \mathbb{E}_\lambda \left|\left\langle t, [\omega(x) - \mathbb{E}_\lambda \omega(x)]x \right\rangle\right|^3.$$

Proposition 2.9. *Under the assumptions of Theorem 2.1,* $L_{\lambda_n} = O\left(\frac{1}{\sqrt{n_1}}\right)$.

Proof. For all $x \in X$, let $\overline{\omega}(x) := \omega(x) - \mathbb{E}_\lambda \omega(x)$. Using the fact that $\Gamma_\lambda^{1/2}$ is symmetric and the Cauchy-Schwarz inequality, notice that we have for all $t \in \mathbb{R}^2$,

$$\sum_{x\in X} \mathbb{E}_\lambda |\langle t, \overline{\omega}(x)\cdot x\rangle|^3 = \sum_{x\in X} \mathbb{E}_\lambda \left|\langle \Gamma_\lambda^{1/2} t, \overline{\omega}(x)\cdot \Gamma_\lambda^{-1/2} x\rangle\right|^3$$
$$\leq \|\Gamma_\lambda^{1/2} t\|^3 \sum_{x\in X} \|\Gamma_\lambda^{-1/2} x\|^3 \mathbb{E}_\lambda |\overline{\omega}(x)|^3.$$

The bound (2.11) of Proposition 2.7 and Jensen's inequality for the convex function $u \mapsto u^{3/2}$ entail the existence of some constant $C > 0$ such that for all $x \in \mathbb{R}^2$,

$$\|\Gamma_{\lambda_n}^{-1/2} x\|^3 = \Gamma_{\lambda_n}^{-1}(x,x)^{3/2} \leq C\left[\left(\frac{1}{n_1}\right)^{3/2} |x_1|^3 + \left(\frac{n_1}{(n_2)^2}\right)^{3/2} |x_2|^3\right].$$

Considering these two facts, we see that it will be enough to show the existence of two positive constants C_1 and C_2 such that

$$\sum_{x\in X} |x_1|^3 \mathbb{E}_\lambda(|\overline{\omega}(x)|^3) \leq C_1 n_1 \qquad \text{and} \qquad \sum_{x\in X} |x_2|^3 \mathbb{E}_\lambda(|\overline{\omega}(x)|^3) \leq C_2 \frac{(n_2)^3}{(n_1)^2}. \tag{2.14}$$

In order to bound the third absolute moment $\mathbb{E}_\lambda(|\overline{\omega}(x)|^3)$, we first compute the fourth moment

$$\mathbb{E}_\lambda(|\overline{\omega}(x)|^4) = \frac{e^{-\langle\lambda,x\rangle}(1 + 7e^{-\langle\lambda,x\rangle} + e^{-2\langle\lambda,x\rangle})}{(1 - e^{-\langle\lambda,x\rangle})^4} \leq \frac{9e^{-\langle\lambda,x\rangle}}{(1 - e^{-\langle\lambda,x\rangle})^4}.$$

Reminding that $\mathbb{E}_\lambda(|\overline{\omega}(x)|^2) = e^{-\langle\lambda,x\rangle}/(1-e^{-\langle\lambda,x\rangle})^2$, and using the Cauchy-Schwarz inequality,

$$\mathbb{E}_\lambda(|\overline{\omega}(x)|^3) \leq \sqrt{\mathbb{E}_\lambda(|\overline{\omega}(x)|^2)\mathbb{E}_\lambda(|\overline{\omega}(x)|^4)} \leq \frac{3e^{-\langle\lambda,x\rangle}}{(1 - e^{-\langle\lambda,x\rangle})^3}.$$

For the first bound of (2.14), we can thus write

$$\sum_{x\in X} |x_1|^3 \mathbb{E}_\lambda(|\overline{\omega}(x)|^3) \leq 3\sum_{x\in X} \frac{|x_1|^3 e^{-\langle\lambda,x\rangle}}{(1 - e^{-\langle\lambda,x\rangle})^3} \leq \frac{3}{(1-e^{-\alpha})^3} \sum_{x_1\geq 1} |x_1|^3 e^{-\alpha x_1} \sum_{x_2 \geq 1} e^{-\beta x_2}.$$

By (2.8), we have $(1 - e^{-\alpha_n}) \asymp 1$ and

$$\sum_{x_1\geq 1} |x_1|^3 e^{-\alpha_n x_1} \asymp e^{-\alpha_n} \asymp \frac{(n_1)^2}{n_2}, \qquad \sum_{x_2\geq 1} e^{-\beta_n x_2} \asymp \frac{1}{\beta_n} \asymp \frac{n_2}{n_1}.$$

Therefore, the first part of (2.14) follows for some positive constant C_1. The second part is obtained similarly from

$$\sum_{x_1\geq 1} e^{-\alpha_n x_1} \asymp e^{-\alpha_n} \asymp \frac{(n_1)^2}{n_2} \quad \text{and} \quad \sum_{x_2\geq 1} |x_2|^3 \frac{e^{-\beta_n x_2}}{(1-e^{-\beta_n x_2})^3} \asymp \frac{1}{\beta_n^4} \asymp \left(\frac{n_2}{n_1}\right)^4. \qquad \square$$

2.6.3 The decrease condition on the characteristic function

We finally check that the last condition of Proposition 2.11 is satisfied. Consider the ellipse $\mathcal{E}_\lambda$ defined by

$$\mathcal{E}_\lambda := \left\{ t \in \mathbb{R}^2 : \|\Gamma_\lambda^{1/2} t\| \leq \frac{1}{4L_\lambda} \right\} = \Gamma_\lambda^{-1/2}\left(\left\{ u \in \mathbb{R}^2 : \|u\| \leq \frac{1}{4L_\lambda} \right\} \right),$$

where L_λ is the Lyapunov ratio previously defined in Subsection 2.6.2.

Proposition 2.10. *Under the assumptions of Theorem 2.1,*

$$\sup_{t\in[-\pi,\pi]^2\setminus\mathcal{E}_{\lambda_n}} \left|\mathbb{E}_{\lambda_n}(e^{i\langle t,N\rangle})\right| = O\left(\frac{1}{n_2\sqrt{n_1}}\right).$$

Proof. Let us write $\varphi_\lambda(t) = \mathbb{E}_\lambda(e^{i\langle t,N\rangle})$ for $t \in \mathbb{R}^2$, the characteristic function of N. Observe that the following elementary inequality holds for all complex number z with modulus $|z| < 1$:

$$\left|\frac{1-|z|}{1-z}\right| \leq \exp\{\mathfrak{Re}(z) - |z|\}. \tag{2.15}$$

Applying (2.15) with $z = e^{-\langle \lambda - it, x\rangle}$ for all $x \in X$, we obtain

$$|\varphi_\lambda(t)| = \prod_{x\in X}\left|\frac{1-e^{-\langle\lambda,x\rangle}}{1-e^{-\langle\lambda-it,x\rangle}}\right| \leq \exp\left\{\mathfrak{Re}\left(\sum_{x\in X} e^{-\langle\lambda-it,x\rangle}\right) - \sum_{x\in X} e^{-\langle\lambda,x\rangle}\right\}.$$

Since $\mathfrak{Re}(z) \leq |z|$ for all complex number z, we deduce that

$$|\varphi_\lambda(t)| \leq \exp\left\{\frac{1}{|e^\alpha - e^{it_1}|}\frac{1}{|e^\beta - e^{it_2}|} - \frac{1}{e^\alpha - 1}\frac{1}{e^\beta - 1}\right\}. \tag{2.16}$$

Let us now describe the set $[-\pi,\pi]^2 \setminus \mathcal{E}_n$. By the inequality (2.10) of Proposition 2.7, we know that there exists a positive constant C such that for all $t \in \mathbb{R}^2$,

$$\|\Gamma_{\lambda_n}^{1/2} t\| = \sqrt{\Gamma_{\lambda_n}(t,t)} \leq C \max\left\{\sqrt{n_1}|t_1|, \frac{n_2}{\sqrt{n_1}}|t_2|\right\}$$

Since $L_\lambda = O(n_1^{-1/2})$, we can find some constant $c > 0$ such that for all $t \in \mathbb{R}^2$, the condition $t \notin \mathcal{E}_n$ implies

$$|t_1| \geq c \qquad \text{or} \qquad |t_2| \geq c\,\frac{n_1}{n_2}. \tag{2.17}$$

In particular, it is enough to bound $|\varphi_\lambda(t)|$ on $\{c \le |t_1| \le \pi\}$ and $\{c\frac{n_1}{n_2} \le |t_2| \le \pi\}$. Without loss of generality, we can assume that $c < \pi$. Let us begin with the case $c \le |t_1| \le c$. It is easy to check that $|e^\alpha - e^{it_1}| \ge |e^\alpha - e^{ic}| \ge (e^\alpha - 1)$ and $|e^\beta - e^{it_2}| \ge (e^\beta - 1)$. Also,

$$\frac{1}{e^{\beta_n}-1} \asymp \frac{n_2}{n_1} \qquad \text{and} \qquad \frac{1}{e^{\alpha_n}-1} - \frac{1}{|e^{\alpha_n}-e^{ic}|} \asymp \frac{(n_1)^2}{n_2}.$$

Hence there exists $C_1 > 0$ such that $|\varphi_\lambda(t)| \le \exp\{-C_1 n_1\}$ uniformly on $\{c < |t_1| \le \pi\}$. We use the same method to bound φ_λ in the domain $\{c\frac{n_1}{n_2} \le |t_2| \le \pi\}$, starting with the inequalities $|e^\alpha - e^{it_1}| \ge (e^\alpha - 1)$, $|e^\beta - e^{it_2}| \ge |e^\beta - e^{ic\beta}|$, and the estimates

$$\frac{1}{e^{\alpha_n}-1} \asymp \frac{(n_1)^2}{n_2} \qquad \text{and} \qquad \frac{1}{e^{\beta_n}-1} - \frac{1}{|e^{\beta_n}-e^{ic\beta_n}|} \asymp \frac{n_2}{n_1}.$$

Thus there exists $C_2 > 0$ such that $|\varphi_\lambda(t)| \le \exp\{-C_2 n_1\}$ uniformly on $\{c\frac{n_1}{n_2} \le |t_2| \le \pi\}$.

Therefore, the existence of a positive constant C such that $|\varphi_n(t)| \le e^{-Cn_1}$ for all $t \in [-\pi,\pi]^2 \setminus \mathcal{E}_n$ follows. This implies the announced result because $n_1 \to +\infty$ and $\log n_2 = o(n_1)$ under the assumptions of Theorem 2.1. □

2.6.4 Proof of the main theorem

We give a proof of Theorem 2.1 in the case $X = \mathbb{N}^2$ of partitions whose parts have non-zero components. Let $(n(k))_k$ be a sequence of vectors in $\mathbb{N}^2$ satisfying the assumptions of the theorem and consider the sequence of parameters $\lambda_k = (\alpha_k, \beta_k)$, taken as the unique solutions of the implicit equations (2.7). Propositions 2.6, 2.8, 2.9 and 2.10 show that, for the rate

$$a_k := \frac{1}{n_2(k)\sqrt{n_1(k)}},$$

all the assumptions of Proposition 2.11 are satisfied. Therefore, there is a local limit theorem of rate a_k for the random variable N under $\mathbb{P}_{\lambda_k}$. In particular,

$$\mathbb{P}_{\lambda_k}(N = n(k)) = \frac{1}{2\pi\sqrt{\det\Gamma_{\lambda_k}}} \exp\left\{-\frac{1}{2}\left\|\Gamma_{\lambda_k}^{-\frac{1}{2}}(n(k) - \mathbb{E}_{\lambda_k}N)\right\|^2\right\} + O(a_k).$$

Remember that we chose the parameters in Section 2.5 to ensure $\|n(k) - \mathbb{E}_{\lambda_k}N\| = O((n_1)^2/n_2)$. By the bound (2.11) of Proposition (2.7) and the Cauchy-Schwarz inequality, we see therefore that $\|\Gamma_{\lambda_k}^{-1/2}(n(k) - \mathbb{E}_{\lambda_k}N)\|$ tends to 0. We then deduce from Proposition 2.6 and $\sqrt{n_1} \to +\infty$ that

$$\mathbb{P}_{\lambda_k}(N = n(k)) \sim \frac{1}{2\pi}\frac{\beta_k^2}{\sqrt{\Delta(\alpha_k)}},$$

which, together with the equality (2.4) and the estimate $\log Z_{\lambda_k} = \frac{\Phi(\alpha_k)}{\beta_k} - \frac{\Psi(\alpha_k)}{2} + o(1)$ following from Proposition 2.3, implies

$$p_{\mathbb{N}^2}(n(k)) \sim \frac{\beta_k^2}{2\pi} \frac{e^{-\frac{1}{2}\Psi(\alpha_k)}}{\sqrt{\Delta(\alpha_k)}} \exp\left\{\alpha_k n_1(k) + \beta_k n_2(k) + \frac{\Phi(\alpha_k)}{\beta_k}\right\}.$$

We finally use the implicit equations (2.7) to simplify this expression.

2.7 A framework for local limit theorems

The aim of this section is to provide a general framework as well as mild conditions under which local limit theorems hold for sums of independent random lattice vectors. We focus on Berry-Esseen-like estimates where existence of third moments is assumed and rates of convergence are established. The conditions need to be flexible enough to handle the strong anisotropy that occurs in our problem. Note that this framework also works in the settings of Báez-Duarte, Sinaĭ, Bogachev and Zarbaliev [7, 44, 17].

Let J be some countable set. Let $\{\xi_j\}_{j\in J}$ be the canonical process on $(\mathbb{Z}^d)^J$ and consider a sequence of product probability measures $(\mathbb{P}_k)$ on the product space $(\mathbb{Z}^d)^J$ such that

$$\sup_k \sum_{j\in J} \mathbb{E}_k \|\xi_j\|^2 < \infty.$$

This condition implies that for all k, the series $\sum_j \xi_j$ converges $\mathbb{P}_k$-almost surely to a random vector S. Moreover, the random vector S has a finite expectation $m_k = \mathbb{E}_k S$ as well as a finite covariance matrix $\Gamma_k = \mathbb{E}_k[(S-m_k)(S-m_k)^\top]$. Let σ_k^2 be the smallest eigenvalue of Γ_k. We make the assumption that Γ_k is *non degenerate* (at least for every k large enough), which is equivalent to $\sigma_k > 0$ so that it has a unique symmetric positive-definite square root $\Gamma_k^{1/2}$, and we write $\Gamma_k^{-1/2}$ for its inverse. Let $g_d(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}\|x\|^2}$ denote the density of the standard normal distribution in $\mathbb{R}^d$.

Definition. *Let (a_k) be a sequence of positive numbers tending to 0. The sequence $(\mathbb{P}_k)$ satisfies a (Gaussian)* local limit theorem with rate a_k *if*

$$\limsup_{k\to+\infty} \sup_{n\in\mathbb{Z}^d} \frac{1}{a_k} \left| \mathbb{P}_k(S=n) - \frac{g_d(\Gamma_k^{-\frac{1}{2}}(n-m_k))}{\sqrt{\det \Gamma_k}} \right| < \infty.$$

We will give simple sufficient conditions for a local limit theorem to hold when the existence of third moments is assumed, that is

$$\sup_k \sum_{j\in J} \mathbb{E}_k \|\xi_j\|^3 < \infty.$$

Under this assumption we associate to each measure $\mathbb{P}_k$ a scale-independent quantity L_k

analogous to the Lyapunov ratio [16, p. 59]:

$$L_k := \sup_{t\in\mathbb{R}^d\setminus\{0\}} \frac{1}{\|\Gamma_k^{1/2}t\|^3} \sum_{j\in J} \mathbb{E}_k|\langle t, \xi_j - \mathbb{E}_k\xi_j\rangle|^3.$$

Finally, we consider the ellipsoid $\mathcal{E}_k$ defined by

$$\mathcal{E}_k := \left\{ t \in \mathbb{R}^d : \|\Gamma_k^{1/2}t\| \le \frac{1}{4L_k} \right\} = \Gamma_k^{-1/2}\left(\left\{ u \in \mathbb{R}^d : \|u\| \le \frac{1}{4L_k} \right\} \right).$$

The following proposition gives three conditions on the product distributions $\mathbb{P}_k$ that entail a local limit theorem with given speed of convergence. Notice that, at least in the one-dimensional *i.i.d.* case, there is no loss in the rate of convergence (consider for example a sequence of independent Bernoulli variables with parameter $0 < p < \frac{1}{2}$).

Proposition 2.11. *Let (a_k) be a sequence of positive numbers tending to* 0 *such that*

$$\frac{1}{\sigma_k\sqrt{\det\Gamma_k}} = O(a_k), \quad \frac{L_k}{\sqrt{\det\Gamma_k}} = O(a_k), \quad \sup_{t\in[-\pi,\pi]^d\setminus\mathcal{E}_k} \left|\mathbb{E}_k(e^{i\langle t,S\rangle})\right| = O(a_k).$$

Then, the sequence $(\mathbb{P}_k)$ satisfies a local limit theorem with rate a_k.

Proof. We resort to Fourier analysis in order to bound the quantity

$$D_k = (2\pi)^d \sup_{n\in\mathbb{Z}^d} \left| \mathbb{P}_k(S=n) - \frac{g_d(\Gamma_k^{-\frac{1}{2}}(n-m_k))}{\sqrt{\det\Gamma_k}} \right|.$$

The strategy of the proof is to compare the distribution of normalized random vector S under the measure $\mathbb{P}_k$ with the normal distribution $\mathcal{N}(m_k,\Gamma_k)$. This is achieved by comparing their respective characteristic functions. Let φ_k be the characteristic function of S under the measure $\mathbb{P}_k$. By definition, we have for all $t \in \mathbb{R}^d$,

$$\varphi_k(t) = \mathbb{E}_k[e^{i\langle t,S\rangle}] = \sum_{n\in\mathbb{Z}^d} \mathbb{P}_k(S=n)\, e^{i\langle t,n\rangle}.$$

The probabilities $\mathbb{P}_k(S=n)$ for $n \in \mathbb{Z}^d$ thus appear as the Fourier coefficients of the periodic function φ_k. In particular, we have an inversion formula:

$$\forall n \in \mathbb{Z}^d, \qquad \mathbb{P}_k(S=n) = \frac{1}{(2\pi)^d} \int_T \varphi_k(t)\, e^{-i\langle t,n\rangle}\, dt,$$

the integral being taken over $T = [-\pi,\pi]^d$.

Now, consider the lattice random vector $Y_k = \Gamma_k^{-1/2}(S - m_k)$. It has zero mean and it is normalized so that its covariance matrix is the identity matrix. Let ψ_k denote the characteristic function of Y_k. By definition, we have $\psi_k(t) = \mathbb{E}_k(e^{i\langle t,Y_k\rangle})$ for all $t \in \mathbb{R}^d$.

Notice that the functions φ_k and ψ_k are linked together by the identity $\psi_k(\Gamma_k^{1/2}t) = \varphi_k(t)e^{-i\langle t,m_k\rangle}$. Hence for every $n \in \mathbb{Z}^d$, one has

$$\mathbb{P}_k(S = n) = \frac{1}{(2\pi)^d} \int_T \psi_k(\Gamma_k^{1/2}t)\, e^{-i\langle t,n-m_k\rangle}\, dt. \tag{2.18}$$

We turn to the second term in D_k, corresponding to the density of the normal distribution $\mathcal{N}(m_k,\Gamma_k)$. The Fourier inversion formula yields for all $n \in \mathbb{Z}^d$,

$$\frac{g_d(\Gamma_k^{-1/2}(n-m_k))}{\sqrt{\det \Gamma_k}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\|\Gamma_k^{1/2}t\|^2} e^{-i\langle t,n-m_k\rangle}\, dt, \tag{2.19}$$

so that equations (2.18) and (2.19) imply together that

$$D_k = \sup_{n\in\mathbb{Z}^d} \left| \int_T \psi_k(\Gamma_k^{1/2}t)\, e^{-i\langle t,n-m_k\rangle}\, dt - \int_{\mathbb{R}^d} e^{-\frac{1}{2}\|\Gamma_k^{1/2}t\|^2} e^{-i\langle t,n-m_k\rangle}\, dt \right|.$$

We split the domain of integration according to the partition $(T \setminus \mathcal{E}_k) \cup (T \cap \mathcal{E}_k) \cup (\mathbb{R}^d \setminus (T \cup \mathcal{E}_k))$ of $\mathbb{R}^d$ and we use the triangular inequality:

$$D_k \le \int_{T\setminus\mathcal{E}_k} |\psi_k(\Gamma_k^{1/2}t)|\, dt + \int_{T\cap\mathcal{E}_k} \left|\psi_k(\Gamma_k^{1/2}t) - e^{-\frac{1}{2}\|\Gamma_k^{1/2}t\|^2}\right| dt + \int_{\mathbb{R}^d\setminus(T\cap\mathcal{E}_k)} e^{-\frac{1}{2}\|\Gamma_k^{1/2}t\|^2}\, dt.$$

Because of the assumption on $\varphi_k(u) = \psi_k(t)$ in the bounded domain $T \setminus \mathcal{E}_k$, the contribution of the first term of the right-hand side is $O(a_k)$. The two other terms are respectively handled by Lemma 2.12 and Lemma 2.13 below. □

Lemma 2.12 (Central approximation)**.** *Under the assumption $L_k = O(a_k\sqrt{\det \Gamma_k})$ of Proposition 2.11,*

$$\limsup_{k\to\infty} \frac{1}{a_k} \int_{\mathcal{E}_k} \left|\psi_k(\Gamma_k^{1/2}t) - e^{-\frac{1}{2}\|\Gamma_k^{1/2}t\|^2}\right| dt < \infty.$$

Proof. After the substitution $u = \Gamma_k^{1/2}t$, and because $\|u\|^3 e^{-\frac{1}{3}\|u\|^2}$ is integrable on $\mathbb{R}^d$, we see that we need only prove the following inequality in the domain $\|u\| \le \frac{1}{4}L_k^{-1}$:

$$\left|\psi_k(u) - e^{-\frac{1}{2}\|u\|^2}\right| \le 16L_k\|u\|^3 e^{-\frac{1}{3}\|u\|^2}, \tag{2.20}$$

We now turn to the proof of (2.20). For all $j \in J$, let ξ_j' be an independent copy of ξ_j. Then $\xi_j - \xi_j'$ has zero mean, its second moments are twice those of the centered random variable $\overline{\xi}_j := \xi_j - \mathbb{E}_k\xi_j$, and $\mathbb{E}_k|\langle t, \xi_j - \xi_j'\rangle|^3 \le 8\mathbb{E}_k|\langle t, \overline{\xi}_j\rangle|^3$. Using $|\mathbb{E}_k[e^{i\langle t,\overline{\xi}_j\rangle}]|^2 = \mathbb{E}_k[e^{i\langle t,\xi_j-\xi_j'\rangle}]$ and the classical Taylor expansion estimate of the characteristic function for $\xi_j - \xi_j'$, we thus have for all $t \in \mathbb{R}^d$,

$$|\mathbb{E}_k[e^{i\langle t,\overline{\xi}_j\rangle}]|^2 \le 1 - \frac{2}{2!}\mathbb{E}_k|\langle t, \overline{\xi}_j\rangle|^2 + \frac{8}{3!}\mathbb{E}_k|\langle t, \overline{\xi_j}\rangle|^3.$$

Since $1-x \le e^{-x}$ for all $x \in \mathbb{R}$, we deduce

$$|\mathbb{E}_k[e^{i\langle t,\overline{\xi}_j\rangle}]| \le \exp\left\{-\frac{1}{2}\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^2 + \frac{2}{3}\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^3\right\}, \tag{2.21}$$

so that, for all $u = \Gamma_k^{1/2}t$ satisfying $\|u\| \le \frac{1}{4}L_k^{-1}$, the definitions of Γ_k and L_k imply

$$|\psi_k(u)|^2 = \prod_{j\in J}|\mathbb{E}_k[e^{i\langle t,\overline{\xi}_j\rangle}]|^2 \le \exp\left\{-\|\Gamma_k^{1/2}t\|^2 + \frac{4}{3}L_k\|\Gamma_k^{1/2}t\|^3\right\} \le \exp\left\{-\frac{2}{3}\|u\|^2\right\}. \tag{2.22}$$

Let us begin with the case $\frac{1}{2}L_k^{-1/3} \le \|u\| \le \frac{1}{4}L_k^{-1}$. In this domain, $\|u\| \ge \frac{1}{2}L_k^{-1/3}$ implies $L_k\|u\|^3 \ge 8$. Using $|\psi_k(u) - e^{-\frac{1}{2}\|u\|^2}| \le |\psi_k(u)| + e^{-\frac{1}{2}\|u\|^2}$ and $e^{-\frac{1}{2}\|u\|^2} \le e^{-\frac{1}{3}\|u\|^2}$, we see that (2.20) holds.

We continue with the remaining case: $\|u\| \le \frac{1}{4}L_k^{-1}$ and $\|u\| \le \frac{1}{2}L_k^{-1/3}$. For all $j \in J$, let $v_j(t) = \exp\{-\frac{1}{2}\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^2\}$ and $w_j(t) = \mathbb{E}_k[e^{i\langle t,\overline{\xi}_j\rangle}]$. By (2.21) we see that $0 < v_j(t) < |w_j(t)|$, so the following elementary inequality holds:

$$\left|\prod_{j\in J} w_j(t) - \prod_{j\in J} v_j(t)\right| \le \left(\prod_{j\in J}|w_j(t)|\right)\sum_{j\in J}\frac{|w_j(t)-v_j(t)|}{v_j(t)}. \tag{2.23}$$

We proved in (2.22) that the product in the right-hand side of (2.23) is bounded by $\exp\{-\frac{1}{3}\|u\|^2\}$. By Jensen's inequality and the definition of L_k, the condition $\|u\| \le \frac{1}{2}L_k^{-1/3}$ implies that

$$v_j(t) \ge \exp\left\{-\frac{1}{2}\left(\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^3\right)^{2/3}\right\} \ge \exp\left\{-\frac{1}{2}(L_k\|u\|^3)^2\right\} \ge \exp\left\{-\frac{1}{8}\right\} > \frac{1}{2}.$$

We estimate the summand in the right-hand side of (2.23) using Taylor expansions of $w_j(t)$ and $v_j(t)$. Since $v_j(t) \ge 1/2$,

$$\begin{aligned}\frac{|w_j(t)-v_j(t)|}{v_j(t)} &\le 2\left|w_j(t) - 1 + \frac{1}{2}\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^2\right| + 2\left|v_j(t) - 1 + \frac{1}{2}\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^2\right| \\ &\le \frac{1}{3}\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^3 + \frac{1}{4}\left(\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^2\right)^2 \le \frac{1}{3}\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^3 + \frac{1}{4}(\mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^3)^{4/3} \\ &\le \mathbb{E}_k|\langle t,\overline{\xi}_j\rangle|^3.\end{aligned}$$

Summing up for all $j \in J$, we obtain $|\psi_k(u) - e^{-\frac{1}{2}\|u\|^2}| \le L_k\|u\|^3 e^{-\frac{1}{3}\|u\|^2}$ and (2.20) follows. □

Lemma 2.13 (Tails completion). *Under the assumptions $L_k = O(a_k\sqrt{\det\Gamma_k})$ and $\sigma_k^{-1} = O(a_k\sqrt{\det\Gamma_k})$ of Proposition 2.11,*

$$\limsup_{k\to\infty}\frac{1}{a_k}\int_{\mathbb{R}^d\setminus(T\cap\mathcal{E}_k)} e^{-\frac{1}{2}\|\Gamma_k^{1/2}t\|^2}\,dt < \infty. \tag{2.24}$$

Proof. The domain of integration splits into $\mathbb{R}^d \setminus (T \cap \mathcal{E}_k) = (\mathbb{R}^d \setminus \mathcal{E}_k) \cup (\mathbb{R}^d \setminus T)$ and we deal separately with the two sub-domains, based on the inequality

$$\int_{\mathbb{R}^d \setminus \mathcal{E}_k} e^{-\frac{1}{2}\|\Gamma_k^{1/2} t\|^2}\, dt \le \frac{1}{\sqrt{\det \Gamma_k}} \left(\int_{\{\|u\| > \frac{1}{4} L_k^{-1}\}} e^{-\frac{1}{2}\|u\|^2}\, du + \int_{\{\Gamma_k^{-1/2} u \notin T\}} e^{-\frac{1}{2}\|u\|^2}\, du \right). \tag{2.25}$$

Let us begin with the first summand. After the polar substitution $(r, \hat{u}) \in (0, +\infty) \times \mathbb{S}^{d-1} \mapsto u = r\hat{u}$, where $\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$, we see that it is proportional (up to the surface area of $\mathbb{S}^{d-1}$) to

$$\int_{r > \frac{1}{4} L_k^{-1}} r^{d-1} e^{-\frac{1}{2} r^2}\, dr \le 4 L_k \int_0^\infty r^d e^{-\frac{1}{2} r^2}\, dr.$$

Since the latter integral is finite and $L_k = O(a_k \sqrt{\det \Gamma_k})$, the first summand of (2.25) yields a finite contribution in (2.24).

In order to deal with the second summand, let us remark that $\Gamma_k^{-1/2} u \notin T$ implies $\|u\| > \sigma_k \pi$ so that the rest of the proof is entirely similar to the first part, except that it uses the assumption on σ_k^{-1}. □

Chapter 3

Lattice convex chains in the plane

A detailed combinatorial analysis of planar lattice convex polygonal lines is presented. This makes it possible to answer an open question of Vershik regarding the existence of a limit shape when the number of vertices is constrained. The method which is used leads also to an asymptotic equivalent formula for the number of convex chains joining two distant points, which involves the non-trivial zeros of the zeta function and establishes a connection with the Riemann Hypothesis.

3.1 Introduction

In 1926, Jarník found an equivalent of the maximal number of integral points that a portion of length n of the graph of a strictly convex function can interpolate. He obtained an explicit constant times $n^{2/3}$. This work was at the origin of many works of Diophantine analysis, and we refer the reader to the papers of W. Schmidt [43] and Bombieri and Pila [18] for more recent results, discussions and open questions on this subject. One may slightly change Jarník's framework, and consider the set of integral points which are interpolated by the graph on $[0,n]$ of an increasing and strictly convex function satisfying $f(0) = 0$ and $f(n) = n$. It turns out that this question is related to another family of works we shall discuss now.

In 1979, Arnold [4] considered the question of the number of equivalence classes of convex lattice polygons having a given integer as area (we say that two polygons having their vertices on $\mathbb{Z}^2$ are equivalent if one is the image of the other by an automorphism of $\mathbb{Z}^2$). Later, Vershik changed the constraint in this question and raised the question of the number, and typical shape, of convex lattice polygons included in a large box $[-n,n]^2$. In 1994, three different solutions to this problem were found by Bárány [8], Vershik [53] and Sinaĭ [44]. Namely, they proved that:

(a) The number of convex polygonal chains with vertices in $(\mathbb{Z}\cap[0,n])^2$ and joining $(0,0)$ to (n,n) is equal to $\exp[3(\zeta(3)/\zeta(2))^{1/3}n^{2/3}(1+o(1))]$, when n goes to infinity.

(b) The number of vertices constituting a typical line is equivalent, when n goes to infinity, to $\frac{n^{2/3}}{(\zeta^2(3)\zeta(2))^{1/3}}$.

(c) The limit shape of a typical convex polygonal line is the arc of a parabola, which maximizes the affine perimeter.

Note that the approach of Sinaĭwas recently made rigorous and extended by Bogachev and Zarbaliev [17].

Later, Vershik and Zeitouni [56] proved, for a class of analogous problems, a large deviation principle involving the affine perimeter of the line. Finally, Acketa and Žunić, while considering the maximal number of vertices for a lattice polygon included in a square, proved shortly after in [1] the analog of Jarník's result, namely that the largest number of vertices for an increasing convex chain on $\mathbb{Z}_+^2$ of Euclidean length n is asymptotically equivalent to $3\left(\frac{n}{\pi}\right)^{2/3}$.

The nature of these results shows that this problem is related to both affine differential geometry and geometry of numbers. Indeed, the parabola found as limit shape coincides with the convex curve inside the square having the largest affine perimeter. Furthermore, the appearance of the values of the Riemann zeta function shows the arithmetic aspects of the problem. One could show indeed that if the lattice $\mathbb{Z}^2$ was replaced by a Poisson Point Process having intensity one (which can be thought as the most isotropic "lattice" one can imagine), the constants $(\zeta^2(3)\zeta(2))^{-1/3} \approx 0.749$ and $3(\zeta(3)/\zeta(2))^{1/3} \approx 2.702$ would be merely raised respectively to 1 and 3 in probability.

Our overall strategy is to use Sinaĭ's statistical mechanics description of the problem. In this model, we are able to establish a very fruitful contour-integral representation of the logarithmic partition function in terms of Riemann's and Barnes' zeta functions. This gives a deeper explanation of the link observed above.

3.1.1 Main results

Our aim in this paper is to improve the three results **(a)**,**(b)**,**(c)** described above. In particular, we shall address the following natural extension of **(c)** which appears as an open question in Vershik's 1994 article:

> *Theorem 3.1 shows how the number of vertices of a typical polygonal line grows. However, one can consider some other fixed growth, say, $\sqrt{n}$, and look for the limit shapes for uniform distributions connected with this growth [...]*

One of our results is that, not only there still exists a limit shape when the number of vertices is constrained, but also the parabolic limit shape is actually universal for all growth rates. The following theorem is a consequence of Theorem 3.3 of Section 3.4 and Theorem 3.5 of Section 3.5.

Theorem. *The Hausdorff distance between a random convex chain on $(\frac{1}{n}\mathbb{Z} \cap [0,1])^2$ joining $(0,0)$ to $(1,1)$ with at most k vertices, and the arc of parabola*

$$\{(x,y) \in [0,1]^2 : \sqrt{y} + \sqrt{1-x} = 1\},$$

converges in probability to 0 *when both* n *and* k *tend to* $+\infty$.

The proof of this theorem requires a detailed combinatorial analysis of convex chains with a constrained number of vertices. This is the purpose of Theorem 3.1 in Section 3.3 which generalizes point **(b)**. We obtain, for any positive number $\mathbf{c}$, a logarithmic equivalent of the number of lines having roughly $\mathbf{c}\,n^{2/3}$ vertices. This question is reminiscent of other ones considered, for instance, by Erdősand Lehner [23], Arratia and Tavaré [5], or Vershik and Yakubovich [55] who were studying combinatorial objects (integer partitions, permutations, polynomials over finite field, Young tableaux, etc.) having a specified number of summands (according to the setting, we call summands, cycles, irreducible divisors, etc.).

The method we use emphasizes the connection of the combinatorial analysis with the zeros of the zeta function. We show in Section 3.8 how the classical theory of the Riemann zeta function leads to an asymptotic equivalent of the number of convex chains, improving point **(a)**. The result can be informally stated as follows.

Theorem. *The number* $p(n)$ *of lattice convex chains in* $[0,n]^2$ *from* $(0,0)$ *to* (n,n) *satisfies*

$$p(n) \sim \frac{e^{-2\zeta'(-1)}}{(2\pi)^{7/6}\sqrt{3}\kappa^{1/18}n^{17/18}} \exp\left[3\kappa^{1/3}n^{2/3} + \sum_{\rho} \frac{\Gamma(\rho)\zeta(\rho+1)\zeta(\rho-1)}{\zeta'(\rho)} \left(\frac{n}{\kappa}\right)^{\rho/3}\right]$$

where $\kappa = \zeta(3)/\zeta(2)$ *and the summation is taken over all zeros of* ζ *in the critical strip.*

A straightforward corollary of this theorem is that, under the assumption that Riemann's Hypothesis is true, the sum of the series inside the exponential is of order roughly $n^{1/6}$ and is oscillating.

3.1.2 Organization of the paper

In Section 3.3, we detail the combinatorial aspect of the result of [8], [53], [44] by proving Theorem 3.1. Following Sinaĭ's approach, the method, borrowed from classical ideas of statistical physics, relies on the introduction of a grand canonical ensemble which endows the considered combinatorial object with a parametrized probability measure. Then, the strategy consists in calibrating the parameters of the probability in order to fit with the constraints one has to deal with. Namely, in our question, it turns out that one can add one parameter in Sinaĭ's probability distribution that makes it possible to take into account, not only the location of the extreme point of the chain but also the number of vertices it contains. In this model, we are able to establish a contour-integral representation of the logarithmic partition function in terms of Riemann's and Barnes' zeta functions. The residue analysis of this representation leads to precise estimates of this function as well as of its derivatives, which correspond to the moments of the random variables of interest such as the position of the terminal point and the number of vertices of the chain. Using a local limit theorem, we finally obtain the asymptotic behavior of the number of lines having $\mathbf{c}\,n^{2/3}$ vertices in terms of the polylogarithm functions $\mathrm{Li}_1, \mathrm{Li}_2, \mathrm{Li}_3$. We also obtain an asymptotic formula for the number of lines having a number k of vertices satisfying $\log n \ll k \ll n^{2/3}$.

In Section 3.4, we derive results about the limit shape of lines having a fixed number of vertices $k \gg \log n$, answering the question of Vershik in a wide range.

In Section 3.5, we extend the results about combinatorics and limit shape beyond $\log n$. The approach here is radically different and more elementary, but limited to $k \ll n^{1/3}$. It relies on the comparison with a continuous setting which has been studied by Bárány [12] and Bárány, Rote, Steiger, Zhang [15].

In Section 3.6, we go back to Jarník's original problem. In addition to Jarník's result that we recover, we give the asymptotic number of chains, typical number of vertices, and limit shape, which is an arc of a circle, in this different framework.

Furthermore, one may mix both types of conditions and the statistical physical method still applies. In Section 3.7, we obtain, for the convex lines joining $(0,0)$ to (n,n) and having a given total length, a continuous family of convex limit shapes that interpolates the diagonal of the square and the two sides of the square, going through the above arc of parabola and arc of circle.

Section 3.8 is devoted to a derivation of the second theorem above about the number of convex chains.

3.2 A one-to-one correspondence

We start this paper by reminding the correspondence between finite convex polygonal chains issuing from 0 whose vertices define increasing sequences in both coordinates and finite distributions of multiplicities on the set of pairs of coprime positive integers.

More precisely, let Π denote the set of finite planar convex polygonal chains Γ issuing from 0 such that the vertices of Γ are points of the lattice $\mathbb{Z}^2$ and the angle between each side of Γ and the horizontal axis is in the interval $[0, \pi/2]$. Now consider the set $\mathbb{X}$ of all vectors $x = (x_1, x_2)$ whose coordinates are coprime positive integers including the pairs $(0,1)$ and $(1,0)$. Sinaĭ observed that the space Π admits a simple alternative description in terms of distributions of multiplicities on $\mathbb{X}$.

Lemma 3.1 (Sinaĭ's correspondence[44])**.** *The space Π is in one-to-one correspondence with the space Ω of nonnegative integer-valued functions $x \mapsto \omega(x)$ on $\mathbb{X}$ with finite support (that is $\omega(x) \neq 0$ for only finitely many $x \in \mathbb{X}$).*

The inverse map $\Omega \to \Pi$ corresponds to the following simple construction: for a given multiplicity distribution $\omega \in \Omega$ and for all $\theta \in [0, \infty]$, let us define

$$X_i^\theta(\omega) := \sum_{\substack{(x_1,x_2)\in\mathbb{X} \\ x_2 \le \theta x_1}} \omega(x) \cdot x_i, \qquad i \in \{1, 2\}. \tag{3.1}$$

When θ ranges over $[0, \infty]$, the function $\theta \mapsto X^\theta(\omega) = (X_1^\theta(\omega), X_2^\theta(\omega))$ takes a finite number of values which are points of the lattice quadrant $\mathbb{Z}_+^2$. These points are in convex position since we are adding vectors in increasing slope order. The convex polygonal curve $\Gamma \in \Pi$ associated to ω is simply the linear interpolation of these points starting from $(0,0)$.

Remark. *This correspondence is a discrete analogue of the Gauss-Minkowski transformation which was used by Vershik and Zeitouni [56].*

3.3 A detailed combinatorial analysis

For every $n = (n_1, n_2) \in \mathbb{Z}_+^2$ and $k \in \mathbb{Z}_+$, define $\Pi(n;k)$ the subset of Π consisting of polygonal chains $\Gamma \in \Pi$ with endpoint n and having k edges, and denote by $p(n;k) := |\Pi(n;k)|$ its cardinality. The restriction of Sinaĭ's correspondence (see Lemma 3.1) to the subspace $\Pi(n;k)$ induces a bijection with the subset $\Omega(n;k)$ of Ω consisting of multiplicity distributions $\omega \in \Omega$ such that the "observables"

$$X_1(\omega) := \sum_{x \in \mathbb{X}} \omega(x) \cdot x_1, \quad X_2(\omega) := \sum_{x \in \mathbb{X}} \omega(x) \cdot x_2, \quad K(\omega) := \sum_{x \in \mathbb{X}} \mathbf{1}_{\{\omega(x) > 0\}}$$

are respectively equal to n_1, n_2 and k. Notice that $X_1 = X_1^\infty$ and $X_2 = X_2^\infty$ with the notations of the previous section.

Out first theorem gives the asymptotic exponential behavior of $p(n;k)$ in terms of the functions $\mathbf{c}$ and $\mathbf{e}$ defined for all $\lambda \in (0, +\infty)$ by

$$\mathbf{c}(\lambda) = \frac{\lambda \mathrm{Li}_2(1-\lambda)}{1-\lambda} \times \frac{1}{\zeta(2)^{1/3}(\zeta(3) - \mathrm{Li}_3(1-\lambda))^{2/3}},$$

$$\mathbf{e}(\lambda) = 3\left(\frac{\zeta(3) - \mathrm{Li}_3(1-\lambda)}{\zeta(2)}\right)^{1/3} - \frac{\lambda \ln(\lambda) \mathrm{Li}_2(1-\lambda)}{1-\lambda} \times \frac{1}{\zeta(2)^{1/3}(\zeta(3) - \mathrm{Li}_3(1-\lambda))^{2/3}}.$$

Theorem 3.1. *Suppose that $|n|$ and k tend to $+\infty$ such that $n_1 \asymp n_2$ and $\log|n|$ is asymptotically negligible compared to k.*

— *If there exists $\lambda \in (0, +\infty)$ such that $k \sim \mathbf{c}(\lambda)(n_1 n_2)^{1/3}$, then*

$$\log p(n;k) \sim \mathbf{e}(\lambda)(n_1 n_2)^{1/3}.$$

— *If k is asymptotically negligible compared to $(n_1 n_2)^{1/3}$, then*

$$p(n;k) = \left(\frac{n_1 n_2}{k^3}\right)^{k+o(k)}.$$

Remark. *It will appear in the core of the proof that one cannot obtain additional terms in the expansion of $\log p(n;k)$ without strong knowledge of the localization of the zeros of Riemann's zeta function. We make this more explicit in the last section of the paper.*

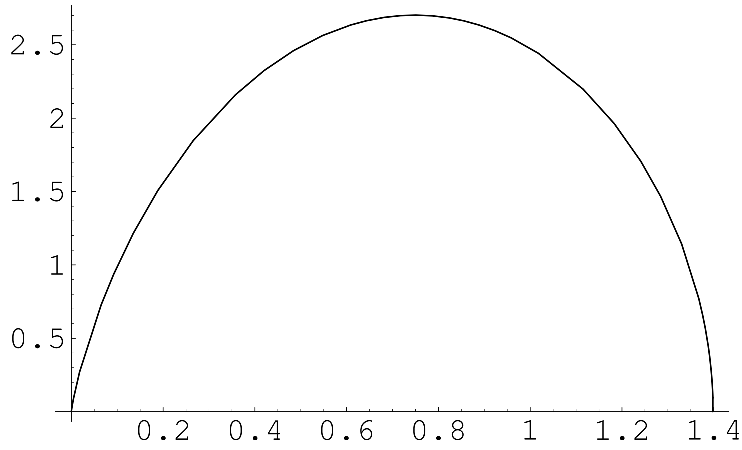

Figure 3.1 – Distribution of the number of vertices of a random convex chain represented by the of $(\mathbf{c}(\lambda), \mathbf{e}(\lambda))$. The point of maximal $\mathbf{e}$-coordinate corresponds to typical chains. The point of maximal $\mathbf{c}$-coordinate corresponds to chains with a maximal number of vertices.

Remark. *The function $\mathbf{e}(\lambda)$ is maximal for $\lambda = 1$, that is to say when there is no penalization. The corresponding coefficients are*

$$\mathbf{c}(1) = \frac{1}{(\zeta(2)\zeta(3)^2)^{1/3}}, \quad \mathbf{e}(1) = 3\left(\frac{\zeta(3)}{\zeta(2)}\right)^{1/3},$$

which recovers the results of [53, 44, 8].

Remark. *As a byproduct of Theorem 3.1, one can deduce the asymptotic behavior of the maximal number of integral points that an increasing convex function satisfying $f(0) = 0$ and $f(n) = n$ can interpolate. This question and its counterpart, concerning the maximal convex lattice polygons inscribed in a convex set was solved by Acketa and Žunić [1] who proved that $M(n) := \max_{\Gamma\in\Pi(n,n)} \left|\Gamma \cap \mathbb{Z}^2\right| \sim 3\left(\pi^{-1}n\right)^{2/3}$.*

Starting from Theorem 3.1, the proof goes as follows. We first notice that $\mathbf{e}(\lambda)$ vanishes when λ goes to infinity. In the same time, $\mathbf{c}(\lambda) \sim \frac{-\mathrm{Li}_2(1-\lambda)}{\zeta(2)^{1/3}(-\mathrm{Li}_3(1-\lambda))^{2/3}}$ which tends to $3\pi^{-2/3}$. Since $\mathbf{e}(\lambda)$ remain strictly positive, we get $\liminf M(n)n^{-2/3} \geq \frac{3}{\pi^{2/3}}$ Now, let $\varepsilon > 0$, and suppose $\limsup M(n)n^{-2/3} \geq \frac{3(1+2\varepsilon)}{\pi^{2/3}}$. Then, for arbitrary large n, there is a chain $\Gamma \in \Pi(n,n)$ having cardinality $[(1+\varepsilon)3(n/\pi)^{2/3}]$. By choosing $k = 3(\pi^{-1}n)^{2/3}$ vertices on this chain, we get already a subset of $\Pi(n;k)$ whose cardinality is $e^{cn^{2/3}}$ with $c > 0$. This enters in contradiction with the fact that $\lim_{\lambda\to\infty} \mathbf{e}(\lambda) = 0$.

3.3.1 Modification of Sinaĭ's model and proof of Theorem 3.1

Let $\mathbb{X} = \{(x_1, x_2) \in \mathbb{Z}_+^2 : \gcd(x_1, x_2) = 1\}$. For any $\omega \in \Omega := \mathcal{F}(\mathbb{X}, \mathbb{Z}_+)$, let $K(\omega)$ be the number of $x \in \mathbb{X}$ such that $\omega(x) > 0$, that is to say

$$K(\omega) := \sum_{x\in\mathbb{X}} \mathbf{1}_{\{\omega(x)>0\}}.$$

For all $\lambda > 0$ and for every couple of parameters $\beta = (\beta_1, \beta_2) \in (0, +\infty)^2$, we endow Ω with the Gibbs measure defined for $\omega \in \Omega$ by

$$\begin{aligned}\mathbb{P}_{\beta,\lambda}(\omega) &:= \frac{1}{Z(\beta,\lambda)} \exp\left[-\sum_{x\in\mathbb{X}} \omega(x)\, \beta\cdot x\right] \lambda^{K(\omega)} \\ &= \frac{1}{Z(\beta,\lambda)} e^{-\beta_1 X_1(\omega)} e^{-\beta_2 X_2(\omega)} \lambda^{K(\omega)},\end{aligned}$$

where the *partition function* $Z(\beta, \lambda)$ is chosen as the normalization constant

$$Z(\beta,\lambda) = \sum_{n\in\mathbb{Z}_+^2} \sum_{k\geq 1} p(n;k)\, e^{-\beta\cdot n} \lambda^k. \tag{3.2}$$

Note that $Z(\beta, \lambda)$ is finite for all values of the parameters $(\beta, \lambda) \in (0, +\infty)^3$. Indeed, if we denote by $p(n) = \sum_{k\geq 1} p(n;k)$ the total number of convex chains of Π with end point $n = (n_1, n_2)$ and M_n the maximal number of edges of such a chain, the following bound holds:

$$Z(\beta,\lambda) \leq \sum_{n\in\mathbb{Z}_+^2} p(n) \max(1,\lambda)^{M_n} e^{-\beta\cdot n}.$$

We use now the results of [8, 53, 44] according to which $\log p(n) = O(|n|^{2/3})$ and of [1] where Acketa and Žunićhave proven that $M_n = O(|n|^{2/3})$. We will use in the sequel the additional remark that $Z(\beta, \lambda)$ is an analytic function of λ for all $\beta > 0$.

Taking $\lambda = 1$, the probability $\mathbb{P}_{\beta,\lambda}$ is nothing but the two-parameter probability distribution introduced by Sinaĭ [44]. Under the measure $\mathbb{P}_{\beta,\lambda}$, the variables $(\omega(x))_{x\in\mathbb{X}}$ are still independent, as in Sinaĭ's framework, but follow a geometric distribution only for $\lambda = 1$. In the general case, the measure $\mathbb{P}_{\beta,\lambda}$ is absolutely continuous with respect to Sinaĭ's measure with density proportional to $\lambda^{K(\cdot)}$ and the distribution of $\omega(x)$ is a biased geometric distribution. Loosely speaking, $\mathbb{P}_{\beta,\lambda}$ corresponds to the introduction of a *penalization* of the probability by a factor λ each time a vertex appears.

Since $\mathbb{P}_{\beta,\lambda}(\omega)$ depends only on the values of $X_1(\omega)$, $X_2(\omega)$, and $K(\omega)$, we deduce that the conditional distribution it induces on $\Omega(n_1, n_2; k)$ is uniform. For instance, we have the following formula for all $(\beta, \lambda) \in (0, +\infty)^2 \times (0, +\infty)$ which will be instrumental in the proof:

$$p(n_1, n_2; k) = Z(\beta,\lambda)\, e^{\beta_1 n_1} e^{\beta_2 n_2} \lambda^{-k}\, \mathbb{P}_{\beta,\lambda}[X_1 = n_1, X_2 = n_2, K = k].$$

In order to get a logarithmic equivalent of $p(n_1, n_2; k)$, our strategy is to choose the three parameters so that

$$\mathbb{E}_{\beta,\lambda}[X_1] = n_1, \quad \mathbb{E}_{\beta,\lambda}[X_2] = n_2, \quad \mathbb{E}_{\beta,\lambda}[K] = k.$$

This will indeed lead to an asymptotic equivalent of $\mathbb{P}_{\beta,\lambda}[X_1 = n_1, X_2 = n_2, K = k]$ due to a local limit result. This equivalent having polynomial decay, it will not interfere with the estimation of $\log p(n_1, n_2; k)$. Together with the analysis of the partition function, this local limit result will constitute the heart of the proof.

3.3.2 Estimates of the logarithmic partition function and its derivatives

We need in the following, the analogue to the Barnes bivariate zeta function defined for $\beta = (\beta_1, \beta_2) \in (0, +\infty)^2$ by

$$\zeta_2^*(s;\beta) := \sum_{x\in\mathbb{X}} (\beta_1 x_1 + \beta_2 x_2)^{-s},$$

this series being convergent for $\mathfrak{Re}(s) > 2$. The following preliminary lemma gives useful properties of this function. This will be done by expressing this function in terms of the Barnes zeta function $\zeta_2(s, w; \beta)$ which is defined by analytic continuation of the series

$$\zeta_2(s,w;\beta) = \sum_{n\in\mathbb{Z}_+^2} (w + \beta_1 n_1 + \beta_2 n_2)^{-s}, \qquad \mathfrak{Re}(s) > 2, \mathfrak{Re}(w) > 0.$$

It is well known that $\zeta_2(s, w; \beta)$ has a meromorphic continuation to the complex s-plane with simple poles at $s = 1$ and 2, and that the residue at $s = 2$ is simply $(\beta_1\beta_2)^{-1}$. In the next lemma, we derive the relation between ζ_2 and ζ_2^*, and we also establish an explicit meromorphic continuation of ζ_2 to the half-plane $\mathfrak{Re}(s) > 1$ in order to obtain later polynomial bounds for $|\zeta_2^*(s)|$ as $|\mathfrak{Im}(s)| \to +\infty$.

Lemma 3.2. *The functions $\zeta_2(s, w; \beta)$ and $\zeta_2^*(s; \beta)$ have a meromorphic continuation to the complex plane.*

(i) The meromorphic continuation of $\zeta_2(s, w; \beta)$ to the half-plane $\mathfrak{Re}(s) > 1$ is given by

$$\begin{aligned}\zeta_2(s,w;\beta) = {} & \frac{1}{\beta_1\beta_2}\frac{w^{-s+2}}{(s-1)(s-2)} + \frac{(\beta_1+\beta_2)w^{-s+1}}{2\beta_1\beta_2(s-1)} + \frac{w^{-s}}{4} \\ & - \frac{\beta_2}{\beta_1}\int_0^{+\infty} \frac{\{y\}-\frac12}{(w+\beta_2 y)^s}\,dy - \frac{\beta_1}{\beta_2}\int_0^{+\infty} \frac{\{x\}-\frac12}{(w+\beta_1 x)^s}\,dx \\ & - s\frac{\beta_2}{2}\int_0^{+\infty} \frac{\{y\}-\frac12}{(w+\beta_2 y)^{s+1}}\,dy - s\frac{\beta_1}{2}\int_0^{+\infty} \frac{\{x\}-\frac12}{(w+\beta_1 x)^{s+1}}\,dx \\ & + s(s+1)\beta_1\beta_2 \int_0^{+\infty}\int_0^{+\infty} \frac{(\{x\}-\frac12)(\{y\}-\frac12)}{(w+\beta_1 x+\beta_2 y)^{s+2}}\,dx\,dy.\end{aligned}$$

(ii) The meromorphic continuation of $\zeta_2^*(s;\beta)$ is given for all $s \in \mathbb{C}$ by

$$\zeta_2^*(s;\beta) = \frac{1}{\beta_1^s} + \frac{1}{\beta_2^s} + \frac{\zeta_2(s, \beta_1+\beta_2;\beta)}{\zeta(s)}.$$

Proof of (i). Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. We apply the Euler-Maclaurin formula to the partial summation defined by $F(x) = \sum_{n_2 \geq 0} (w + \beta_1 x + \beta_2 n_2)^{-s}$, leading to

$$\sum_{n_1 \geq 1} F(n_1) = \int_0^\infty F(x)\,dx - \frac{F(0)}{2} + \int_0^\infty (\{x\} - \frac{1}{2}) F'(x)\,dx.$$

We use again the Euler-Maclaurin formula for each of the summations in n_2. □

Proof of (ii). Let us express $\zeta_2^*(s;\beta)$ in terms of $\zeta_2(s, \beta_1+\beta_2;\beta)$ for all s with real part $\Re(s) > 2$. The result will follow from the analytic continuation principle. By definition of $\zeta_2^*(s;\beta)$,

$$\zeta_2^*(s;\beta) - \frac{1}{\beta_1^s} - \frac{1}{\beta_2^s} = \sum_{x_1,x_2 \geq 1} \frac{1}{(\beta_1 x_1 + \beta_2 x_2)^s} 1_{\gcd(x_1,x_2)=1}.$$

Using the classical Möbius function $\mu(d)$ taking values in $\{-1,0,1\}$ and the Möbius inversion formula (see [30]), we obtain

$$1_{\gcd(x_1,x_2)=1} = \sum_{d=1}^\infty \mu(d) 1_{d|x_1} 1_{d|x_2},$$

so we can write the latter expression as

$$\sum_{d \geq 1} \sum_{x_1,x_2 \geq 1} \frac{\mu(d)}{(\beta_1 x_1 + \beta_2 x_2)^s} 1_{d|x_1} 1_{d|x_2} = \sum_{d \geq 1} \frac{\mu(d)}{d^s} \sum_{x_1',x_2' \geq 1} \frac{1}{(\beta_1 x_1 + \beta_2 x_2)^s}.$$

Finally, the classical formula

$$\sum_{d=1}^\infty \frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)}$$

which holds for all s having real part $\Re(s) > 1$ implies (ii). □

Now we make the connection between these zeta functions and the logarithmic partition function of our modified Sinaĭ's model. Let us recall that the polylogarithm function $\mathrm{Li}_s(z)$, also known as Jonquière function, is defined for all complex number $s \in \mathbb{C}$ by analytic continuation of the series

$$\mathrm{Li}_s(z) = \sum_{k=1}^\infty \frac{z^k}{k^s}, \qquad |z| < 1.$$

For our purpose, the continuation given by the Bose-Einstein integral

$$\mathrm{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z t^{s-1}}{e^t - z} dt$$

for $\mathfrak{Re}(s) > 0$ and $z \in \mathbb{C} \setminus [1, +\infty)$ will be sufficient.

Lemma 3.3. *Let $c > 2$. For all parameters $(\beta, \lambda) \in (0, +\infty)^2 \times (0, +\infty)$,*

$$\log Z(\beta, \lambda) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} (\zeta(s+1) - \mathrm{Li}_{s+1}(1-\lambda))\zeta_2^*(s;\beta)\Gamma(s) ds.$$

Proof. Given the product form of the distribution $\mathbb{P}_{\beta,\lambda}$, we see that the random variables $\omega(x)$ for $x \in \mathbb{X}$ are mutually independent. Moreover, the marginal distribution of $\omega(x)$ is a biased geometric distribution. It is absolutely continuous with respect to the geometric distribution of parameter $e^{-\beta\cdot x}$ with density proportional to $k \mapsto \lambda^{1_{k>0}}$. In other words, for all $k \in \mathbb{Z}_+$,

$$\mathbb{P}_{\beta,\lambda}[\omega(x) = k] = Z_x(\beta, \lambda)^{-1} e^{-k\beta\cdot x} \lambda^{1_{k>0}}$$

where the normalization constant $Z_x(\beta, \lambda) = 1 + \lambda \dfrac{e^{-\beta\cdot x}}{1 - e^{-\beta\cdot x}}$ is easily computed. We can now deduce the following product formula for the partition function:

$$Z(\beta, \lambda) = \prod_{x\in\mathbb{X}} Z_x(\beta, \lambda) = \prod_{x\in\mathbb{X}} \left(1 + \lambda \frac{e^{-\beta\cdot x}}{1 - e^{-\beta\cdot x}}\right).$$

For now, we assume that $\lambda \in (0, 1)$. Taking the logarithm of the product above

$$\begin{aligned}\log Z(\beta, \lambda) &= \sum_{x\in\mathbb{X}} \log\left(1 + \lambda \frac{e^{-\beta\cdot x}}{1 - e^{-\beta\cdot x}}\right)\\ &= \sum_{x\in\mathbb{X}} \log(1 - (1-\lambda)e^{-\beta\cdot x}) - \sum_{x\in\mathbb{X}} \log(1 - e^{-\beta\cdot x})\\ &= \sum_{x\in\mathbb{X}} \sum_{r\geq 1} \frac{1 - (1-\lambda)^r}{r} e^{-r\beta\cdot x}.\end{aligned}$$

Now we use the fact that the Euler gamma function $\Gamma(s)$ and the exponential function are related through Mellin's inversion formula

$$e^{-z} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} ds,$$

for all $c > 0$ and $z \in \mathbb{C}$ with positive real part. Choosing $c > 2$ so that the series and the integral all converge and applying the Fubini theorem, this yields

$$\begin{aligned}\log Z(\beta,\lambda) &= \frac{1}{2i\pi}\sum_{x\in\mathbb{X}}\sum_{r\geq 1}\int_{c-i\infty}^{c+i\infty}\frac{1-(1-\lambda)^r}{r}r^{-s}(\beta\cdot x)^{-s}\Gamma(s)ds\\ &= \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}(\zeta(s+1)-\mathrm{Li}_{s+1}(1-\lambda))\zeta_2^*(s;\beta)\Gamma(s)\,ds.\end{aligned}$$

The lemma is proven for all $\lambda\in(0,1)$. The extension to $\lambda>0$ will now result from analytic continuation. We already noticed that the left hand term is analytic in λ for all fixed β. Proving the analyticity of the right hand term requires only to justify the absolute convergence of the integral on the vertical line. From Lemma 3.2, we know that $\zeta_2^*(c+i\tau;\beta)$ is polynomially bounded as $|\tau|$ tends to infinity. Taking $s=c-1+i\tau$, successive integrations by parts of the formula

$$(\zeta(s+1)-\mathrm{Li}_{s+1}(1-\lambda))\Gamma(s+1)=\lambda\int_0^{\infty}\frac{e^x x^s}{(e^x-1)(e^x-1+\lambda)}dx$$

show for all integer $N>0$, there exists a constant $C_N>0$ such that, uniformly in τ,

$$\left|(\zeta(s+1)-\mathrm{Li}_{s+1}(1-\lambda))\Gamma(s+1)\right|\leq\frac{C_N\lambda}{(1+|\tau|)^N}. \tag{3.3}$$

□

Finally, the next Lemma makes use of the contour integral representation of $\log Z(\beta,\lambda)$ to derive at the same time an asymptotic formula for each one of its derivatives.

Lemma 3.4. *Let $(p,q_1,q_2)\in\mathbb{Z}_+^3$. For all $\varepsilon>0$, there exists $C>0$ such that*

$$\left|\left[\lambda\frac{\partial}{\partial\lambda}\right]^p\left[\frac{\partial}{\partial\beta_1}\right]^{q_1}\left[\frac{\partial}{\partial\beta_2}\right]^{q_2}\left(\log Z(\beta,\lambda)-\frac{\zeta(3)-\mathrm{Li}_3(1-\lambda)}{\zeta(2)\beta_1\beta_2}\right)\right|\leq\frac{C\lambda}{|\beta|^{\kappa}}$$

with $\kappa=q_1+q_2+1+\varepsilon$, uniformly in the region $\{(\beta,\lambda):\varepsilon<\frac{\beta_1}{\beta_2}<\frac{1}{\varepsilon}$ and $0<\lambda<\frac{1}{\varepsilon}\}$.

Proof. Lemma 3.3 provides an integral representation of the logarithmic partition function $\log Z(\beta,\lambda)$. We will use the residue theorem to shift the contour of integration from the vertical line $\mathfrak{Re}(s)=3$ to the line $\mathfrak{Re}(s)=1+\varepsilon$. Lemma 3.2 shows that the function $M(s):=(\zeta(s+1)-\mathrm{Li}_{s+1}(1-\lambda)))\zeta_2^*(s;\beta)\Gamma(s)$ is meromorphic in the strip $1<\mathfrak{Re}(s)<3$ with a single pole at $s=2$, where the residue is given by

$$\frac{\zeta(3)-\mathrm{Li}_3(1-\lambda)}{\zeta(2)}\cdot\frac{1}{\beta_1\beta_2}$$

From the inequality (3.3), Lemma 3.2 and the fact that $|\zeta(s)|$ has no zero with $\mathfrak{Re}(s)>1$, we see that $M(s)$ vanishes uniformly in $1+\varepsilon\leq\mathfrak{Re}(s)\leq 3$ when $|\mathfrak{Im}(s)|$ tends to $+\infty$. By the residue theorem,

$$\log Z(\beta,\lambda) = \frac{\zeta(3) - \mathrm{Li}_3(1-\lambda))}{\zeta(2)\beta_1\beta_2} + \frac{1}{2i\pi}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} M(s)\,ds. \tag{3.4}$$

From the Leibniz rule applied in the formula of Lemma 3.2 (i), we obtain directly the meromorphic continuation of $\frac{\partial^{q_1}}{\partial\beta_1^{q_1}}\frac{\partial^{q_2}}{\partial\beta_2^{q_2}}\zeta_2(s,\beta_1+\beta_2;\beta)$ in the half-plane $\mathfrak{Re}(s) > 1$. We also obtain the existence of a constant $C > 0$ such that

$$\left|\left[\frac{\partial}{\partial\beta_1}\right]^{q_1}\left[\frac{\partial}{\partial\beta_2}\right]^{q_2}\zeta_2(1+\varepsilon+i\tau,\beta_1+\beta_2;\beta)\right| \le \frac{C\,|\tau|^{2+q_1+q_2}}{|\beta|^\kappa}$$

with $\kappa = q_1+q_2+1+\varepsilon$. A reasoning similar to the one we have used in order to derive (3.3) shows that for all integers p and $N > 0$, there exists a constant $C_{p,N}$ such that, uniformly in τ,

$$\left|\left[\lambda\frac{\partial}{\partial\lambda}\right]^{p}(\zeta(s+1) - \mathrm{Li}_{s+1}(1-\lambda))\Gamma(s+1)\right| \le \frac{C_{p,N}\,\lambda}{(1+|\tau|)^N}.$$

In order to differentiate both sides of equation (3.4) and permute the partial derivatives and the integral sign, we have to mention the fact that the Riemann zeta function is bounded from below on the line $\mathfrak{Re}(s) = 1+\varepsilon$ and that the derivatives of $\mathrm{Li}_s(1-\lambda)$ with respect to λ are all bounded. This also gives the announced bound on the error term. □

3.3.3 Calibration of the shape parameters

When governed by the Gibbs measure $\mathbb{P}_{\beta,\lambda}$, the expected value of the random vector with components

$$X_1(\omega) = \sum_{x\in\mathbb{X}}\omega(x)x_1, \quad X_2(\omega) = \sum_{x\in\mathbb{X}}\omega(x)x_2, \quad K(\omega) = \sum_{x\in\mathbb{X}}\mathbf{1}_{\{\omega(x)>0\}},$$

is simply given by the logarithmic derivatives of the partition function $Z(\beta,\lambda)$. Remember that we planned to choose λ and β_1,β_2 as functions of $n = (n_1,n_2)$ an k in order for the probability $\mathbb{P}[X_1 = n_1, X_2 = n_2, K = k]$ to be maximal, which is equivalent to $\mathbb{E}(X_1) = n_1$, $\mathbb{E}(X_2) = N_2$ and $\mathbb{E}(K) = k$. We address this question in the next lemma.

Lemma 3.5. *Assume that n_1,n_2,k tend to infinity with $n_1 \asymp n_2$ and $|k| = O(|n|^{2/3})$. There exists a unique choice of $(\beta_1,\beta_2,\lambda)$ as functions of (n,k) such that*

$$\mathbb{E}_{\beta,\lambda}[X_1] = n_1, \quad \mathbb{E}_{\beta,\lambda}[X_2] = n_2, \quad \mathbb{E}_{\beta,\lambda}[K] = k.$$

Moreover, they satisfy

$$n_1 \sim \frac{\zeta(3) - \mathrm{Li}_3(1-\lambda)}{\zeta(2)(\beta_1)^2\beta_2}, \quad n_2 \sim \frac{\zeta(3) - \mathrm{Li}_3(1-\lambda)}{\zeta(2)\beta_1(\beta_2)^2}, \quad k \sim -\frac{\lambda\partial_\lambda \mathrm{Li}_3(1-\lambda)}{\zeta(2)\beta_1\beta_2}. \tag{3.5}$$

If $k = o(|n|^{2/3})$, then λ goes to 0 and the above relations yield

$$\beta_1 \sim \frac{k}{n_1}, \quad \beta_2 \sim \frac{k}{n_2}, \quad \lambda \sim \frac{k^3}{n_1 n_2}.$$

Proof. With the change of variable $\lambda = e^{-\gamma}$, the existence and uniqueness of (β, λ) are equivalent to the fact that the function

$$f\colon (\beta_1, \beta_2, \gamma) \mapsto \beta_1 n_1 + \beta_2 n_2 + \gamma k + \log Z(\beta, e^{-\gamma})$$

has a unique critical point in the open domain $D = (0, +\infty)^2 \times \mathbb{R}$. First observe that f is smooth and strictly convex since its Hessian matrix is actually the covariance matrix of the random vector (X_1, X_2, K). In addition, from the very definition (3.2) of $Z(\beta, \lambda)$, we can see that f converges to $+\infty$ in the neighborhood of any point of the boundary of D as well as when $|\beta_1| + |\beta_2| + |\gamma|$ tends to $+\infty$. The function being continuous in D, this implies the existence of a minimum, which by convexity is the unique critical point (β^*, γ^*) of f.

From now on, we will be concerned and check along the proof that we stay in the regime $\beta_1, \beta_2 \to 0$, $\beta_1 \asymp \beta_2$, and γ bounded from below. From Lemma 3.4, we can approximate f by the simpler function

$$g\colon (\beta_1, \beta_2, \gamma) \mapsto \beta_1 n_1 + \beta_2 n_2 + \gamma k + \frac{\zeta(3) - \mathrm{Li}_3(1-e^{-\gamma})}{\beta_1\beta_2}$$

with $|f(\beta, \gamma) - g(\beta, \gamma)| \le \dfrac{Ce^{-\gamma}}{|\beta|^{3/2}}$ for some constant $C > 0$. The unique critical point $(\tilde\beta, \tilde\gamma)$ of g satisfies

$$n_1 = \frac{\zeta(3) - \mathrm{Li}_3(1-e^{-\tilde\gamma})}{\zeta(2)(\tilde\beta_1)^2\tilde\beta_2}, \quad n_2 = \frac{\zeta(3) - \mathrm{Li}_3(1-e^{-\tilde\gamma})}{\zeta(2)\tilde\beta_1(\tilde\beta_2)^2}, \quad k = -\frac{e^{-\tilde\gamma}\partial_\lambda \mathrm{Li}_3(1-e^{-\tilde\gamma})}{\zeta(2)\tilde\beta_1\tilde\beta_2}.$$

The goal now is to prove that (β^*, γ^*) is close to $(\tilde\beta, \tilde\gamma)$. To this aim, we find a convex neighborhood C of $(\tilde\beta, \tilde\gamma)$ such that $g|_{\partial C} \ge g(\tilde\beta, \tilde\gamma) + \frac{Ce^{-\tilde\gamma}}{\tilde\beta_1\tilde\beta_2}$. In the neighborhood of $(\tilde\beta, \tilde\gamma)$ the expression of the Hessian matrix of g yields $g(\tilde\beta_1 + t_1, \tilde\beta_2 + t_2, \tilde\gamma + u) \ge g(\tilde\beta_1, \tilde\beta_2, \tilde\gamma) + \frac{\tilde C e^{-\tilde\gamma}}{(\tilde\beta_1\tilde\beta_2)^2}(\|t\|^2 + \tilde\beta_1\tilde\beta_2|u|^2)$. Therefore we need only take

$$C = [\tilde\beta_1 - C_1\tilde\beta_1^{5/4}, \tilde\beta_1 + C_1\tilde\beta_1^{5/4}] \times [\tilde\beta_2 - C_2\tilde\beta_2^{5/4}, \tilde\beta_2 + C_2\tilde\beta_2^{5/4}] \times [\tilde\gamma - C_3|\beta|^{1/4}, \tilde\gamma + C_3|\beta|^{1/4}].$$

Therefore, $f|_{\partial C} > f(\tilde{\beta},\tilde{\gamma})$. By convexity of f and C this implies $(\beta^*,\gamma^*) \in C$. Hence

$$\beta_1^* \sim \tilde{\beta}_1, \quad \beta_2^* \sim \tilde{\beta}_2, \quad e^{-\gamma^*} \sim e^{-\tilde{\gamma}},$$

concluding the proof. □

3.3.4 A local limit theorem

In this section, we show that the random vector (X_1, X_2, K) satisfies a local limit theorem when the parameters are calibrated as above. Let $\Gamma_{\beta,\lambda}$ be the covariance matrix under the measure $\mathbb{P}_{\beta,\lambda}$ of the random vector (X_1, X_2, K).

Theorem 3.2 (Local limit theorem). *Let us assume that n_1, n_2, k tend to infinity such that $n_1 \asymp n_2 \asymp |n|$, $\log|n| = o(k)$, and $k = O(|n|^{2/3})$. For the choice of parameters made in Lemma 3.5,*

$$\mathbb{P}_{\beta,\lambda}[X = n, K = k] \sim \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{\det \Gamma_{\beta,\lambda}}}. \tag{3.6}$$

Moreover,

$$\det \Gamma_{\beta,\lambda} \asymp \frac{|n|^4}{k} \tag{3.7}$$

If $k = o(|n|^{2/3})$,

$$\mathbb{P}_{\beta,\lambda}[X = n, K = k] \sim \frac{1}{(2\pi)^{3/2}} \frac{\sqrt{k}}{n_1 n_2} \tag{3.8}$$

This result is actually an application of a more general lemma proven by the first author in [19]*Proposition 7.1. In order to state the lemma, we introduce some notations. Let $\sigma^2_{\beta,\lambda}$ be the smallest eigenvalue of $\Gamma_{\beta,\lambda}$. Introducing $X_{1,x} = \omega(x) \cdot x_1$, $X_{2,x} = \omega(x) \cdot x_2$ and $K_x = 1_{\{\omega(x)>0\}}$ as well as $\overline{X_{1,x}}, \overline{X_{2,x}}, \overline{K_x}$ their centered counterparts, let $L_{\beta,\lambda}$ be the Lyapunov ratio

$$L_{\beta,\lambda} := \sup_{(t_1,t_2,u)\in\mathbb{R}^3} \sum_{x\in\mathbb{X}} \frac{\mathbb{E}_{\beta,\lambda}\left|t_1\overline{X_{1,x}} + t_2\overline{X_{2,x}} + u\overline{K_x}\right|^3}{\Gamma_{\beta,\lambda}(t_1,t_2,u)^{3/2}}.$$

where $\Gamma_{\beta,\lambda}(\cdot)$ stands for the quadratic form canonically associated to $\Gamma_{\beta,\lambda}$. Let $\phi_{\beta,\lambda}(t,u) = \mathbb{E}_{\beta,\lambda}(e^{i(t_1X_1+t_2X_2+uK})$ for all $(t_1,t_2,u) \in \mathbb{R}^3$. Finally, we consider the ellipsoid $\mathcal{E}_{\beta,\lambda}$ defined by

$$\mathcal{E}_{\beta,\lambda} := \left\{(t_1,t_2,u) \in \mathbb{R}^3 : \Gamma_{\beta,\lambda}(t_1,t_2,u) \le (4L_{\beta,\lambda})^{-2}\right\}.$$

The following lemma is a reformulation of Proposition 7.1 in [19]. It gives three conditions on the product distributions $\mathbb{P}_{\beta,\lambda}$ that entail a local limit theorem with given speed of convergence.

Lemma 3.6. *With the notations introduced above, suppose that there exists a family of number $(a_{\beta,\lambda})$ such that*

$$\frac{1}{\sigma_{\beta,\lambda}\sqrt{\det\Gamma_{\beta,\lambda}}} = O(a_{\beta,\lambda}), \tag{3.9}$$

$$\frac{L_{\beta,\lambda}}{\sqrt{\det\Gamma_{\beta,\lambda}}} = O(a_{\beta,\lambda}), \tag{3.10}$$

$$\sup_{(t,u)\in[-\pi,\pi]^3\setminus\mathcal{E}_{\beta,\lambda}} \left|\phi_{\beta,\lambda}(t,u)\right| = O(a_{\beta,\lambda}). \tag{3.11}$$

Then, a local limit theorem holds uniformly for $\mathbb{P}_{\beta,\lambda}$ with rate $a_{\beta,\lambda}$:

$$\sup_{(n,k)\in\mathbb{Z}^3}\left|\mathbb{P}_{\beta,\lambda}[X=n,K=k] - \frac{\exp\left[-\frac{1}{2}\Gamma_{\beta,\lambda}^{-1}\Big((n,k)-\mathbb{E}_{\beta,\lambda}(X,K)\Big)\right]}{(2\pi)^{3/2}\sqrt{\det\Gamma_{\beta,\lambda}}}\right| = O(a_{\beta,\lambda}).$$

When governed by the Gibbs measure $\mathbb{P}_{\beta,\lambda}$, the covariance matrix $\Gamma_{\beta,\lambda}$ of the random vector (X_1,X_2,K) is simply given by the Hessian matrix of the log partition function $\log Z(\beta,\lambda)$. Let $u(\lambda) := (\zeta(3)-\mathrm{Li}_3(1-\lambda))/\zeta(2)$ for $\lambda>0$. Applications of Lemma 3.4 for all $(p,q_1,q_2)\in\mathbb{Z}_+^3$ such that $p+q_1+q_2=2$ imply that this covariance matrix is asymptotically equivalent to

$$\begin{bmatrix}\beta_1\beta_2 & 0 & 0\\ 0 & \beta_1^3\beta_2 & 0\\ 0 & 0 & \beta_1\beta_2^3\end{bmatrix}^{-\frac{1}{2}}\begin{bmatrix}\lambda^2u''(\lambda)+\lambda u'(\lambda) & \lambda u'(\lambda) & \lambda u'(\lambda)\\ \lambda u'(\lambda) & 2u(\lambda) & u(\lambda)\\ \lambda u'(\lambda) & u(\lambda) & 2u(\lambda)\end{bmatrix}\begin{bmatrix}\beta_1\beta_2 & 0 & 0\\ 0 & \beta_1^3\beta_2 & 0\\ 0 & 0 & \beta_1\beta_2^3\end{bmatrix}^{-\frac{1}{2}}.$$

A straightforward calculation shows that this matrix is positive definite for all $\lambda>0$.

Lemma 3.7. *The random vector (X_1,X_2,K) has a covariance matrix $\Gamma_{\beta,\lambda}$ satisfying*

$$\Gamma_{\beta,\lambda}(t,u) \asymp \frac{(n_1)^{5/3}}{(\lambda n_2)^{1/3}}|t_1|^2 + \frac{(n_2)^{5/3}}{(\lambda n_1)^{1/3}}|t_2|^2 + (\lambda n_1 n_2)^{1/3}|u|^2, \qquad |n|\to+\infty.$$

Proof. All the coefficients of the previous matrix $u(\lambda), \lambda u'(\lambda), \lambda^2 u''(\lambda)$ are of order λ in the neighborhood of 0, and the determinant is equivalent to λ^3. Therefore, the eigenvalues are also of order λ. The result follows from the fact that the values of β_1 and β_2 are given by (3.5) and that $\zeta(3)-\mathrm{Li}_3(1-\lambda)\asymp\zeta(2)\lambda$. □

Lemma 3.8. *The Lyapunov coefficient satisfies $L_{\beta,\lambda} = O(\lambda^{-1/6}|n|^{-1/3})$.*

Proof. Using Lemma 3.7, there exists a constant $C>0$ such that

$$L_{\beta,\lambda} \le C\sum_{x\in\mathbb{X}}\left[\frac{\mathbb{E}_{\beta,\lambda}|\overline{X_{1,x}}|^3}{\lambda^{-1/2}}\frac{n_2^{1/2}}{n_1^{5/2}} + \frac{\mathbb{E}_{\beta,\lambda}|\overline{X_{2,x}}|^3}{\lambda^{-1/2}}\frac{n_1^{1/2}}{n_2^{5/2}} + \frac{\mathbb{E}_{\beta,\lambda}|\overline{K_x}|^3}{\lambda^{1/2}(n_1n_2)^{1/2}}\right].$$

Therefore, we need only prove that

$$\sum_{x\in\mathbb{X}} \mathbb{E}_{\beta,\lambda} \left|\overline{K_x}\right|^3 = O(|n|^{2/3}), \qquad \sum_{x\in\mathbb{X}} \mathbb{E}_{\beta,\lambda} \left|\overline{X_{i,x}}\right|^3 = O(|n|^{5/3}).$$

Notice that for a Bernoulli random variable $B(p)$ of parameter p, one has $\mathbb{E}[|B(p)-p|^3] \leq 4(\mathbb{E}[B(p)^3] + p^3) \leq 8p$. This implies

$$\sum_{x\in\mathbb{X}} \mathbb{E}_{\beta,\lambda} \left|\overline{K_x}\right|^3 \leq \sum_{x\in\mathbb{X}} \frac{8\lambda e^{-\beta\cdot x}}{1-(1-\lambda)e^{-\beta\cdot x}} \leq \sum_{x\in\mathbb{X}} \frac{8\lambda e^{-\beta\cdot x}}{1-e^{-\beta\cdot x}} = O(\frac{\lambda}{\beta_1\beta_2}).$$

Similarly, we obtain

$$\sum_{x\in\mathbb{X}} \mathbb{E}_{\beta,\lambda} \left|\overline{X_{1,x}}\right|^3 = O(\frac{\lambda}{\beta_1^4\beta_2}), \quad \sum_{x\in\mathbb{X}} \mathbb{E}_{\beta,\lambda} \left|\overline{X_{2,x}}\right|^3 = O(\frac{\lambda}{\beta_1\beta_2^4}).$$

□

Lemma 3.9. *Condition* (3.11) *of Lemma 3.6 is satisfied. More precisely,*

$$\limsup_{|n|\to+\infty} \sup_{(t,u)\in[-\pi,\pi]^3\setminus\mathcal{E}_{\beta,\lambda}} \frac{1}{\lambda^{1/3}|n|^{2/3}} \log|\phi_n(t,u)| < 0.$$

Proof. From Lemmas 3.7 and 3.8, there exists a constant $c>0$ depending on λ such that for all $n=(n_1,n_2)$ with $|n|$ large enough,

$$[-\pi,\pi]^3 \setminus \mathcal{E}_{\lambda,n} \subset \{(t,u)\in\mathbb{R}^3 : c<|u|\leq\pi \text{ or } c\lambda^{1/3}|n|^{-1/3}<|t|\}.$$

The strategy of the proof is to deal separately with the cases $|u|>c$ and $|t|>c\lambda^{1/3}|n|^{-1/3}$, which requires to find first adequate bounds for $|\phi_n(t,u)|$ in both cases. For all $(t_1,t_2,u)\in\mathbb{R}^3$ and $x\in\mathbb{X}$, let us write $t=(t_1,t_2)$ and $\rho^x=e^{-\beta\cdot x}$. The "partial" characteristic function $\phi_n^x(t,u)=\mathbb{E}[e^{i(t\cdot X_x+uK_x)}]$ is given by

$$\phi_n^x(t,u) = \left(1+\lambda e^{iu}\frac{e^{it\cdot x}\rho^x}{1-e^{it\cdot x}\rho^x}\right)\left(1+\lambda\frac{\rho^x}{1-\rho^x}\right)^{-1},$$

hence a straightforward calculation yields

$$\begin{aligned}\left|\phi_n^x(t,u)\right|^2 &= 1-\frac{\frac{4\lambda\rho^x}{(1-(1-\lambda)\rho^x)^2}\left[\frac{\rho^x(2+(\lambda-2)\rho^x)}{(1-\rho^x)^2}|\sin(\frac{t\cdot x}{2})|^2+|\sin(\frac{t\cdot x+u}{2})|^2-\rho^x|\sin(\frac{u}{2})|^2\right]}{1+\frac{4\rho^x}{(1-\rho^x)^2}|\sin(\frac{t\cdot x}{2})|^2}\\ &\leq \exp\left\{-\frac{\frac{4\lambda\rho^x}{(1-(1-\lambda)\rho^x)^2}\left(2\rho^x|\sin(\frac{t\cdot x}{2})|^2+|\sin(\frac{t\cdot x+u}{2})|^2-\rho^x|\sin(\frac{u}{2})|^2\right)}{1+\frac{4\rho^x}{(1-\rho^x)^2}|\sin(\frac{t\cdot x}{2})|^2}\right\}\end{aligned}$$

Using the law of sines in a triangle with angles $\frac{t\cdot x}{2}$, $\frac{u}{2}$ and $\frac{2\pi - t\cdot x+u}{2}$, we see that the numerator inside the bracket is proportional (with positive constant) to

$$2\rho^x\|a\|^2 + \|b\|^2 - \rho^x\|a+b\|^2$$

where a and b are two-dimensional vectors. Since the real quadratic form $(a_i, b_i) \mapsto 2\rho\, a_i^2 + b_i^2 - \frac{2\rho}{1+2\rho}(a_i+b_i)^2$ is positive for all $\rho \in (0,1)$ and for $i \in \{1,2\}$, we deduce that

$$|\phi_n^x(t,u)| \le \exp\left\{ -\frac{\frac{2\lambda\rho^x}{(1-(1-\lambda)\rho^x)^2}}{1+\frac{4\rho^x}{(1-\rho^x)^2}} \left(\frac{2\rho^x}{1+2\rho^x} - \rho^x \right) \left|\sin(\tfrac{u}{2})\right|^2 \right\} \tag{3.12}$$

for all x such that $\rho_x \le \frac{1}{2}$. In the same way, the positivity of the quadratic form $(a_i, b_i) \mapsto \frac{\rho}{1-\rho}\, a_i^2 + b_i^2 - \rho\,(a_i+b_i)^2$ yields

$$|\phi_n^x(t,u)| \le \exp\left\{ -\frac{\frac{2\lambda\rho^x}{(1-(1-\lambda)\rho^x)^2}}{1+\frac{4\rho^x}{(1-\rho^x)^2}} \left(2\rho^x - \frac{\rho^x}{1-\rho^x} \right) \left|\sin(\tfrac{t\cdot x}{2})\right|^2 \right\} \tag{3.13}$$

for all x such that $\rho_x \le \frac{1}{2}$.

Let us begin with the region $\{(t,u) \in \mathbb{R}^3 : c < |u| \le \pi\}$. In this case $|\sin(\frac{u}{2})|$ is uniformly bounded from below by $|\sin(\frac{c}{2})|$. Hence using (3.12) for the $x \in \mathbb{X}$ such that $\frac{1}{4} < \rho^x \le \frac{1}{3}$ and the bound $|\phi_n^x(t,u)| \le 1$ for all other x, we obtain

$$\log|\phi_n(t,u)| \le -\frac{1}{160} \frac{\lambda|\sin(\frac{c}{2})|^2}{(1+\frac{1}{3}|\lambda-1|)^2} \left| \left\{ x \in \mathbb{X} : \frac{1}{4} < \rho^x \le \frac{1}{3} \right\} \right|.$$

To conclude, let us recall that the number of integral points with coprime coordinates such that $\frac{1}{4} < e^{-\beta\cdot x} \le \frac{1}{3}$ is asymptotically equal to $\frac{1}{\zeta(2)} \frac{\log(4/3)}{2\beta_1\beta_2} \asymp \lambda^{-2/3}|n|^{2/3}$.

We now turn to the region $\{(t,u) \in [-\pi,\pi]^3 : c\lambda^{1/3}|n|^{-1/3} < |t|\}$. Without loss of generality, we can assume $|t_1| > c'\lambda^{1/3}|n|^{-1/3}$ for some universal constant $c' \in (0;c)$. Using the inequality (3.13) for the elements $x \in \mathbb{X}$ such that $\frac{1}{4} < \rho^x \le \frac{1}{3}$ and the bound $|\phi_n^x(t,u)| \le 1$ for all other x, we obtain for all $\varepsilon \in (0,1)$,

$$\log|\phi_n(t,u)| \le -\frac{\varepsilon^2}{64} \frac{\lambda}{(1+\frac{1}{3}|\lambda-1|)^2} \left| \left\{ x \in \mathbb{X} : \frac{1}{4} < \rho^x \le \frac{1}{3} \text{ and } |\sin(\tfrac{t\cdot x}{2})| \ge \varepsilon \right\} \right|.$$

Since the number of $x \in \mathbb{X}$ such that $\frac{1}{4} < e^{-\beta\cdot x} \le \frac{1}{3}$ is asymptotically equal to $\frac{\log(4/3)}{2\zeta(2)\beta_1\beta_2}$, it is enough to prove that we can find ε such that the set of vectors $x \in \mathbb{Z}_+^2$ with $|\sin(\frac{t\cdot x}{2})| < \varepsilon$ has density strictly smaller than $\frac{1}{\zeta(2)}$ in $\{x \in \mathbb{Z}_+^2 : \frac{1}{4} < \rho^x \le \frac{1}{3}\}$. We split up this region according to horizontal lines, that is to say with $\frac{t_2x_2}{2}$ constant. The set $\{x_1 \in \mathbb{R} : |\sin(\frac{t_2x_2}{2} + \frac{t_1x_1}{2})| < \varepsilon\}$ is a periodic union of strips of period $\tau_1 = \frac{2\pi}{t_1} \ge 2$ and width bounded by $4\varepsilon\tau_1$.

Hence the number of $x_1 \in \mathbb{Z}_+$ satisfying this condition and lying in any bounded finite interval I is at most $\left(\frac{|I|}{\tau_1}+2\right)(4\varepsilon\tau_1+1)$. Summing up the contributions of the horizontal lines, this shows the existence of some positive constant $C > 0$ independent of ε such that for all $\varepsilon \in (0,1)$, the number of $x \in \mathbb{Z}_+^2$ satisfying both $\frac{1}{4} < e^{-\beta\cdot x} \leq \frac{1}{3}$ and $|\sin(\frac{t\cdot x}{2})| < \varepsilon$ is bounded by

$$(\tfrac{1}{2}+C\varepsilon)\frac{\log(4/3)}{2\beta_1\beta_2}+C|n|^{1/3}\log|n|.$$

To achieve our goal, we can therefore choose $\varepsilon = \frac{1}{2C}(\frac{1}{\zeta(2)}-\frac{1}{2}) > 0$. □

Proof of Theorem 3.2. We simply check that the hypotheses of Lemma 3.6 are satisfied. From Lemma 3.7, we have $\sigma_{\beta,\lambda}^2 \asymp k$ and $\det(\Gamma_{\beta,\lambda}) \asymp k^{-1}|n|^4$, hence

$$\frac{1}{\sigma_{\beta,\lambda}\sqrt{\det\Gamma_{\beta,\lambda}}} \asymp \frac{1}{|n|^2}.$$

Using in addition Lemma 3.8, we have also

$$\frac{L_{\beta,\lambda}}{\sqrt{\det\Gamma_{\beta,\lambda}}} = O\left(\frac{1}{|n|^2}\right).$$

Finally, Lemma 3.9 shows the existence of some constant $c > 0$ such that for all (n,k) large enough,

$$\sup_{(t,u)\in[-\pi,\pi]^3\setminus\mathcal{E}_{\beta,\lambda}} |\phi_n(t,u)| \leq e^{-ck}$$

Since we have made the assumption $\log|n| = o(k)$, the quantity e^{-ck} is also bounded from above by $|n|^{-2}$. Therefore, all hypotheses of Lemma 3.6 are satisfied. As a consequence, $\mathbb{P}_{\beta,\lambda}$ satisfies a local limit theorem with speed rate $a_{\beta,\lambda} \asymp |n|^{-2}$.

□

3.4 Limit shape

We start by proving the existence of a limit shape in the modified Sinaĭ model, which is the aim of the next two lemmas. The natural normalization for the convex chain is to divide each coordinate by the corresponding expectations for the final point.

The first lemma shows that the arc of parabola is the limiting curve of the expectation of the random convex chain $m_i^\theta(\beta,\lambda) = \mathbb{E}_{\beta,\lambda}[X_i^\theta]$ for $i \in \{1,2\}, \theta \in [0,\infty]$ under the $\mathbb{P}_{\beta,\lambda}$ distribution.

Lemma 3.10. *Suppose that β_1 and β_2 tend to 0 such that $\beta_1 \asymp \beta_2$ and λ is bounded from* *above. Then*

$$\lim_{|\beta|\to 0}\sup_{\theta\in[0,\infty]}\left\|\left[\frac{m_1^\theta(\beta,\lambda)}{m_1^\infty(\beta,\lambda)},\frac{m_2^\theta(\beta,\lambda)}{m_2^\infty(\beta,\lambda)}\right]-\left[\frac{\theta(\theta+2\frac{\beta_1}{\beta_2})}{(\theta+\frac{\beta_1}{\beta_2})^2},\frac{\theta^2}{(\theta+\frac{\beta_1}{\beta_2})^2}\right]\right\|=0.$$

Proof. Since we are dealing with increasing functions, the uniform convergence convergence will follow from the simple convergence. We mimic the proof of Lemma 3.4, except that the domain of summation $\mathbb{X}$ is replaced by the subset of vectors x such that $x_2 \le \theta x_1$. The expectations are given by the first derivatives of the *partial* logarithmic partition function

$$\log Z^\theta(\beta,\lambda)=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}(\zeta(s+1)-\mathrm{Li}_{s+1}(1-\lambda))\zeta_2^{\theta,*}(s)\Gamma(s)\,ds$$

where $\zeta_2^{\theta,*}$ is the restricted zeta function defined by analytic continuation of the series

$$\begin{aligned}\zeta_2^{\theta,*}(s)&=\sum_{\substack{x\in\mathbb{X}\\ x_2\le\theta x_1}}(\beta_1x_1+\beta_2x_2)^{-s}\\ &=\frac{1}{\beta_1^s}+\frac{1_{\{\theta=\infty\}}}{\beta_2^s}+\frac{1}{\zeta(s)}\sum_{\substack{x_1,x_2\ge1\\ x_2\le\theta x_1}}(\beta_1x_1+\beta_2x_2)^{-s}.\end{aligned}$$

The continuation of the underlying restricted Barnes zeta function is obtained using the Euler-Maclaurin formula several times:

$$
\begin{aligned}
\sum_{x_2=1}^{\lfloor\theta x_1\rfloor}(\beta_1x_1+\beta_2x_2)^{-s} &= \int_1^{\lfloor\theta x_1\rfloor}(\beta_1x_1+\beta_2x_2)^{-s}\,dx_2+\frac{(\beta_1x_1+\beta_2)^{-s}}{2}+\frac{(\beta_1x_1+\beta_2\lfloor\theta x_1\rfloor)^{-s}}{2}\\
&\quad -s\beta_2\int_1^{\lfloor\theta x_1\rfloor}(\{x_2\}-\frac{1}{2})(\beta_1x_1+\beta_2x_2)^{-(s+1)}\,dx_2\\
&= \int_1^{\theta x_1}(\beta_1x_1+\beta_2x_2)^{-s}\,dx_2+\frac{(\beta_1x_1+\beta_2)^{-s}}{2}+\frac{(\beta_1x_1+\beta_2\lfloor\theta x_1\rfloor)^{-s}}{2}\\
&\quad -s\beta_2\int_1^{\lfloor\theta x_1\rfloor}(\{x_2\}-\frac{1}{2})(\beta_1x_1+\beta_2x_2)^{-(s+1)}\,dx_2\\
&\quad -\int_{\lfloor\theta x_1\rfloor}^{\theta x_1}(\beta_1x_1+\beta_2x_2)^{-s}\,dx_2\\
&= \int_1^{\theta x_1}(\beta_1x_1+\beta_2x_2)^{-s}\,dx_2+\frac{(\beta_1x_1+\beta_2)^{-s}}{2}+\frac{(\beta_1x_1+\beta_2\lfloor\theta x_1\rfloor)^{-s}}{2}\\
&\quad -s\beta_2\int_1^{\lfloor\theta x_1\rfloor}(\{x_2\}-\frac{1}{2})(\beta_1x_1+\beta_2x_2)^{-(s+1)}\,dx_2\\
&\quad -\int_{\lfloor\theta x_1\rfloor}^{\theta x_1}(\beta_1x_1+\beta_2x_2)^{-s}\,dx_2\\
&= \frac{(\beta_1x_1+\beta_2)^{-s+1}}{\beta_2(s-1)}-\frac{(\beta_1x_1+\beta_2\theta x_1)^{-s+1}}{\beta_2(s-1)}+R(s,x_1,\beta_1,\beta_2,\theta)
\end{aligned}
$$

where

$$
\begin{aligned}
R(s,x_1,\beta_1,\beta_2,\theta) &= \frac{(\beta_1x_1+\beta_2)^{-s}}{2}+\frac{(\beta_1x_1+\beta_2\lfloor\theta x_1\rfloor)^{-s}}{2}\\
&\quad -s\beta_2\int_1^{\lfloor\theta x_1\rfloor}(\{x_2\}-\frac{1}{2})(\beta_1x_1+\beta_2x_2)^{-(s+1)}\,dx_2\\
&\quad -\int_{\lfloor\theta x_1\rfloor}^{\theta x_1}(\beta_1x_1+\beta_2x_2)^{-s}\,dx_2
\end{aligned}
$$

is such that $\sum_{x_1\geq 1}R(s,x_1,\beta_1,\beta_2,\theta)$ converges absolutely for all s with $\mathfrak{Re}(s)>1$. Therefore the latter series defines an holomorphic function in the half-plane $\mathfrak{Re}(s)>1$. Finally,

$$
\begin{aligned}
\sum_{\substack{x_1,x_2\geq 1\\ x_2\leq\theta x_1}}(\beta_1x_1+\beta_2x_2)^{-s} &= \frac{(\beta_1+\beta_2)^{-s+2}}{\beta_1\beta_2(s-1)(s-2)}-\frac{(\beta_1+\theta\beta_2)^{-s+2}}{(\beta_1+\theta\beta_2)\beta_2(s-1)(s-2)}\\
&\quad +\widetilde{R}(s,\beta_1,\beta_2,\theta)
\end{aligned}
$$

where $\widetilde{R}$ is holomorphic in s for $\mathfrak{Re}(s) > 1$. Hence, the residue at $s = 2$ is $\frac{\theta}{\beta_1(\beta_1+\theta\beta_2)}$. Taking the derivatives with respect to β_1 and β_2, we obtain,

$$-\frac{\partial}{\partial\beta_1} \sum_{\substack{x_1,x_2\geq 1 \\ x_2\leq\theta x_1}} (\beta_1 x_1 + \beta_2 x_2)^{-s} = \frac{1}{\beta_1^2\beta_2} \frac{\theta(\theta + 2\frac{\beta_1}{\beta_2})}{(\theta + \frac{\beta_1}{\beta_2})^2} \frac{1}{s-2} + R_1(s, \beta_1, \beta_2, \theta)$$

and similarly

$$-\frac{\partial}{\partial\beta_2} \sum_{\substack{x_1,x_2\geq 1 \\ x_2\leq\theta x_1}} (\beta_1 x_1 + \beta_2 x_2)^{-s} = \frac{1}{\beta_1\beta_2^2} \frac{\theta^2}{(\theta + \frac{\beta_1}{\beta_2})^2} \frac{1}{s-2} + R_2(s, \beta_1, \beta_2, \theta)$$

where both remainder terms R_1 and R_2 are holomorphic in s in the half-plane $\sigma := \mathfrak{Re}(s) > 1$ and are bounded, up to positive constants, by

$$\frac{|s|^2}{\sigma - 1} \min(\beta_1, \beta_2)^{-\sigma-1}.$$

This decrease makes it possible to apply the residue theorem in order to shift to the left the vertical line of integration from $\sigma = 3$ to $\sigma = \frac{3}{2}$. When β_1 and β_2 tend to 0 and $\frac{\beta_1}{\beta_2}$ tends to ℓ, we thus find

$$\mathbb{E}_{\beta,\lambda}[X_1^\theta] = \frac{\zeta(3) - \mathrm{Li}_3(1-\lambda)}{\zeta(2)} \left[\frac{1}{\beta_1^2\beta_2} \frac{\theta(\theta + 2\frac{\beta_1}{\beta_2})}{(\theta + \frac{\beta_1}{\beta_2})^2} + O\left(\frac{1}{|\beta|^{5/2}}\right) \right],$$

$$\mathbb{E}_{\beta,\lambda}[X_2^\theta] = \frac{\zeta(3) - \mathrm{Li}_3(1-\lambda)}{\zeta(2)} \left[\frac{1}{\beta_1^2\beta_2} \frac{\theta^2}{(\theta + \frac{\beta_1}{\beta_2})^2} + O\left(\frac{1}{|\beta|^{5/2}}\right) \right].$$

We obtain the announced result by normalizing these quantities by their limits when θ goes to infinity. □

Lemma 3.11 (Uniform exponential concentration)**.** *Suppose that β_1 and β_2 tend to 0 such that $\beta_1 \asymp \beta_2$ and λ is bounded from above. For all $\eta \in (0,1)$, we have*

$$\mathbb{P}_{\beta,\lambda}\left[\sup_{1\leq i\leq 2} \sup_{\theta\in[0,\infty]} \frac{|X_i^\theta - m_i^\theta(\beta,\lambda)|}{m_i^\infty(\beta,\lambda)} > \eta \right] \leq \exp\left\{ -\frac{c(\lambda)\eta^2}{8\beta_1\beta_2}(1 + o(1)) \right\}.$$

Proof. Fix $i \in \{1,2\}$ and let $M_\theta = X_i^\theta - m_i^\theta(\beta,\lambda)$ for all $\theta \geq 0$. The stochastic process $(M_\theta)_{\theta\geq 0}$ is a $\mathbb{P}_{\beta,\lambda}$-martingale, therefore $(e^{tM_\theta})_{\theta\geq 0}$ is a positive $\mathbb{P}_{\beta,\lambda}$-submartingale for any choice of $t \geq 0$ such that $\mathbb{E}_{\beta,\lambda}[e^{tX_i}]$ is finite. This condition is satisfied when $t < \beta_1$.

Doob's martingale inequality implies for all $\eta > 0$,

$$\mathbb{P}_{\beta,\lambda}\left[\sup_{\theta\in[0,\infty]} M_\theta > \eta\, m_i^\infty(\beta,\lambda)\right] = \mathbb{P}_{\beta,\lambda}\left[\sup_{\theta\in[0,\infty]} e^{tM_\theta} > e^{t\eta m_i^\infty(\beta,\lambda)}\right]$$
$$\leq e^{-t\eta m_i^\infty(\beta,\lambda)}\mathbb{E}_{\beta,\lambda}\left[e^{tM_\infty}\right] = e^{-t(\eta+1)m_i^\infty(\beta,\lambda)}\mathbb{E}_{\beta,\lambda}[e^{tX_i}]$$

For $i = 1$, Lemma 3.4 shows that the logarithm of the right-hand side satisfies

$$-t(1+\eta)m_1^\infty(\beta,\lambda) + \log\frac{Z(\beta_1 - t,\beta_2;\lambda)}{Z(\beta_1,\beta_2;\lambda)} = \frac{c(\lambda)}{\beta_1\beta_2}\left[-\frac{t(1+\eta)}{\beta_1} - 1 + \frac{\beta_1}{\beta_1 - t} + o(1)\right]$$

asymptotically when t and β_1 are of the same order. The same holds for $i = 2$. This is roughly optimized for the choice $t = \beta_i\left(1-(1+\eta)^{-1/2}\right)$, which gives

$$\mathbb{P}_{\beta,\lambda}\left[\sup_{\theta\in[0,\infty]} M_\theta > \eta\, m_i^\infty(\beta,\lambda)\right] \leq \exp\left\{-\frac{2c(\lambda)}{\beta_1\beta_2}\left(1+\frac{\eta}{2}-\sqrt{1+\eta}+o(1)\right)\right\}.$$

When considering the martingale defined by $N_\theta = m_i^\theta(\beta,\lambda) - X_i^\theta$, one obtains with the same method

$$\mathbb{P}_{\beta,\lambda}\left[\sup_{\theta\in[0,\infty]} N_\theta > \eta\, m_i^\infty(\beta,\lambda)\right] \leq \exp\left\{-\frac{2c(\lambda)}{\beta_1\beta_2}\left(1-\frac{\eta}{2}-\sqrt{1-\eta}+o(1)\right)\right\}.$$

Since the previous inequalities hold for both $i \in \{1,2\}$, a simple union bound now yields

$$\mathbb{P}_{\beta,\lambda}\left[\sup_{1\leq i\leq 2}\sup_{\theta\in[0,\infty]}\frac{|X_i^\theta - m_i^\theta(\beta,\lambda)|}{m_i^\infty(\beta,\lambda)} > \eta\right] \leq 4\exp\left\{-\frac{c(\lambda)\eta^2}{8\beta_1\beta_2}(1+o(1))\right\}.$$

□

We introduce the following parametrization of the arc of parabola $\sqrt{y}+\sqrt{1-x} = 1$:

$$x_1(\theta) = \frac{\theta(\theta+2)}{(\theta+1)^2}, \quad x_2(\theta) = \frac{\theta^2}{(\theta+1)^2}, \qquad \theta\in[0,\infty].$$

Theorem 3.3 (Limit shape for numerous vertices). *Assume that $n_1 \asymp n_2 \to +\infty$, and $k = O(|n|^{2/3})$, and $\log|n| = o(k)$. There exists $c > 0$ such that for all $\eta \in (0,1)$,*

$$\mathbb{P}_{n,k}\left[\sup_{1\leq i\leq 2}\sup_{\theta\in[0,\infty]}\frac{|X_i^\theta - x_i(\frac{\beta_2}{\beta_1}\theta)|}{n_i} > \eta\right] \leq \exp\left\{-c\eta^2 k(1+o(1))\right\}.$$

In particular, the Hausdorff distance between a random convex chain on $\frac{1}{n}\mathbb{Z}_+^2$ joining $(0,0)$ to $(1,1)$ with at most k vertices and the arc of parabola $\sqrt{y}+\sqrt{1-x} = 1$ converges in probability *to* 0.

Proof. Using the triangle inequality and Lemma 3.10, we need only prove the analogue of Lemma 3.11 for the uniform probability $\mathbb{P}_{n,k}$. Remind that the measure $\mathbb{P}_{\beta,\lambda}$ conditional on the event $\{X = n, K = k\}$ is nothing but the uniform probability $\mathbb{P}_{n,k}$. Hence for all event E,

$$\mathbb{P}_{n,k}(E) \leq \frac{\mathbb{P}_{\beta,\lambda}(E)}{\mathbb{P}_{\beta,\lambda}(X = n, K = k)}.$$

Applying this with the deviation event above for the parameters (β, λ) defined in Section 3.3.3 and using the Local Limit Theorem 3.2 as well as the concentration bound provided by Lemma 3.11, the right-hand side reads, up to constants,

$$\frac{|n|^2}{\sqrt{k}} \exp\left\{-c\eta^2 k(1 + o(1))\right\}.$$

Since $\log|n| = o(k)$, the result follows. □

3.5 Chains with few vertices

3.5.1 Combinatorial analysis

The previous machinery does not apply in the case of very few vertices but it can be completed by an an elementary approach that we present now which will actually work up to a number of vertices negligible compared to $n^{1/3}$. It is based on the following heuristics: when n tends to $+\infty$ and the number of edges k is very small compared to n, one can expect that choosing an element of $\Pi(n;k)$ at random is somewhat similar to choosing $k-1$ vertices from $[0,1]^2$ in convex position at random. Bárány [12] and Bárány, Rote, Steiger, Zhang [15] proved by two different methods the existence of a parabolic limit shape in this continuous setting. These works are based on Valtr's observation that each convex chain with k edges is associated, by permutation of the edges, to exactly $k!$ increasing North-East polygonal chains with pairwise different slopes.

Our first theorem is the convex-chain analogue to a result of Erdös and Lehner on integer partitions [23]*Theorem 4.1.

Theorem 3.4. *The number of convex chains joining* $(0,0)$ *to* (n,n) *with* k *edges satisfies*

$$|\Pi(n;k)| = \frac{1}{k!}\binom{n-1}{k-1}^2 (1 + o(1))),$$

this formula being valid uniformly in k *for* $k = o(n^{1/2}/(\log n)^{1/4})$.

Proof. Let us start by proving an upper bound. This is done by considering the inequality

$$|\Pi(n;k)| \leq \frac{1}{k!}\binom{n-1}{k-1}^2 + \frac{2}{(k-1)!}\binom{n-1}{k-2}\binom{n-1}{k-1} + \frac{1}{(k-2)!}\binom{n-1}{k-2}^2$$

where the first term bounds the number of convex chains which are associated to strictly North-East chains, the second term bounds the number of convex chains having either a first horizontal vector or a last vertical one, and the third term bounds the numbers of convex chains having both a horizontal and a vertical vector.

We now turn to a lower bound. Let $\{U_1, U_2, \ldots, U_{k-1}\}$ and $\{V_1, V_2, \ldots, V_{k-1}\}$ be two independent uniformly random subsets of $\{1, \ldots, n-1\}$ of size $k-1$ whose elements are indexed in increasing order $U_1 < U_2 < \cdots < U_{k-1}$ and $V_1 < V_2 < \cdots < V_{k-1}$. Let $M_0 = (0,0)$, $M_k = (n,n)$ and $M_i = (U_i, V_i)$ for $1 \le i \le k-1$. Obviously, the polygonal chain $(M_0, M_1, \ldots, M_n)$ has uniform distribution among all increasing polygonal chain from $(0,0)$ to (n,n). We claim that the distribution of $(\overrightarrow{M_0M_1}, \overrightarrow{M_1M_2}, \ldots, \overrightarrow{M_{k-1}M_k})$ conditioned on the event that no two of these vectors are parallel is uniform among the chains of $\Pi(n,k)$ such that no side is parallel to the x-axis or the y-axis. Moreover, since the vectors are exchangeable, the probability that we can find $i < j$ such that $\overrightarrow{M_{i-1}M_i}$ and $\overrightarrow{M_{j-1}M_j}$ are parallel is bounded from above by $\binom{k}{2}$ times the probability that $Y = \overrightarrow{M_0M_1}$ and $Z = \overrightarrow{M_1M_2}$ are parallel. Using the simple estimate

$$\binom{n-1}{k-1} \ge \frac{n^{k-1}}{(k-1)!}(1 - o(1))$$

which is asymptotically true since $k = o(\sqrt{n})$, we find that for all $(y,z) \in (\mathbb{N}^2)^2$, the probability that $Y = y$ and $Z = y$ is

$$\begin{aligned}
\mathbb{P}(Y = y, Z = z) &= \frac{\binom{n-y_1-z_1}{k-3}\binom{n-y_2-z_2}{k-3}}{\binom{n-1}{k-1}^2} \\
&\le \frac{4k^2}{n^2}\left(1 - \frac{y_1+z_1}{n}\right)_+^{k-3}\left(1 - \frac{y_2+z_2}{n}\right)_+^{k-3} \\
&\le \frac{4k^2}{n^2}\exp\left\{-\frac{k-3}{n}(y_1+y_2+z_1+z_2)\right\}.
\end{aligned}$$

We can therefore dominate the probability that Y and Z are parallel by the probability that geometrically distributed random vectors are parallel, which is exactly estimated in the following lemma applied with $\beta = \frac{k}{n}$. In conclusion, the probability that at least two vectors are parallel is bounded by $\frac{k^4}{n^2}\log(n)$ up to a constant. □

Lemma 3.12. *Let Y_1, Y_2, Z_1, Z_2 be independent and identically distributed geometric random variables of parameter $1 - e^{\beta}$ with $\beta > 0$. When β goes to 0, the probability that the vectors $Y = (Y_1, Y_2)$ and $Z = (Z_1, Z_2)$ are parallel is asymptotically equal to*

$$\frac{\beta^2}{\zeta(2)}\log\frac{1}{\beta}.$$

Proof. The probability that Y and Z are parallel is

$$\sum_{x\in\mathbb{X}}\sum_{i,j\geq 1}\mathbb{P}(Y=ix,Z=jx)=(1-e^{-\beta})^4\sum_{x\in\mathbb{X}}\sum_{i,j\geq 1}e^{-\beta(i+j)(x_1+x_2)}.$$

The Mellin transform of the double summation in the right-hand side with respect to $\beta>0$ is well-defined for all $s\in\mathbb{C}$ with $\mathfrak{Re}(s)>2$ and it is equal to

$$\sum_{x\in\mathbb{X}}\sum_{i,j\geq 1}\frac{\Gamma(s)}{(x_1+x_2)^s(i+j)^s}=\frac{\Gamma(s)}{\zeta(s)}(\zeta(s-1)-\zeta(s))^2.$$

Expanding this Mellin transform in Laurent series at the pole $s=2$ of order 2 and using the residue theorem to express the Mellin inverse, one finds

$$\sum_{x\in\mathbb{X}}\sum_{i,j\geq 1}e^{-\beta(i+j)(x_1+x_2)}=\frac{1}{\zeta(2)}\frac{\log\frac{1}{\beta}}{\beta^2}-\frac{C}{\beta^2}+O\left(\frac{1}{\beta}\right),\qquad \text{as } \beta\to 0.$$

where $C=\frac{2\zeta(2)-\zeta'(2)-1-\gamma}{\zeta(2)}\approx 0.471207$. □

3.5.2 Limit shape

Theorem 3.5 (Limit shape for few vertices)**.** *The Hausdorff distance between a random convex chain in* $(\frac{1}{n}\mathbb{Z}\cap[0,1])^2$ *joining* $(0,0)$ *to* $(1,1)$ *having at most k vertices and the arc of parabola* $\sqrt{y}+\sqrt{1-x}=1$ *converges in probability to* 0 *when both n and k tend to* $+\infty$ *with* $k=o(n^{1/3})$.

Proof. Bárány [12] and Bárány, Rote, Steiger, Zhang [15] proved by two different methods the existence of a limit shape in the following continuous setting: if one picks at random $k-1$ points uniformly from the square $[0,1]^2$, then conditional on the event that these points are in convex position, the Hausdorff distance between the convex polygonal chain thus defined and the parabolic arc goes to 0 in probability as k goes to $+\infty$. Our strategy is to show that this result can be extended to the discrete setting $([0,1]\cap\frac{1}{n}\mathbb{Z})^2$ if k is small enough compared to n by using a natural embedding of the discrete model into the continuous model.

For this purpose, we first observe that the distribution of the above continuous model can be described as follows: pick uniformly at random $k-1$ points from both the x-axis and the y-axis, rank them in increasing order and let $0=U_0<U_1<U_2<\cdots<U_{k-1}<U_k=1$ and $0=V_0<V_1<V_2<\cdots<V_{k-1}<V_k=1$ denote this ranking. The points (U_i,V_i) define an increasing North-East polygonal chain joining $(0,0)$ to $(1,1)$. Reordering the segment lines of this chain by increasing slope order, exchangeability arguments show that we obtain a convex chain with k edges that follows the desired distribution. This is analogous to the discrete construction of strictly North-East convex chains from $(0,0)$ to (n,n) that occurs in the proof of Theorem 3.4.

Now, we define the lattice-valued random variables $\tilde{U}_0 \leq \tilde{U}_1 \leq \tilde{U}_2 \leq \cdots \leq \tilde{U}_{k-1} \leq \tilde{U}_k$ and $\tilde{V}_0 \leq \tilde{V}_1 \leq \tilde{V}_2 \leq \cdots \leq \tilde{V}_{k-1} \leq \tilde{V}_k$ by discrete approximation:

$$\begin{cases} \tilde{U}_i \in \frac{1}{n}\mathbb{Z}, & U_i \leq \tilde{U}_i < U_i + \frac{1}{n} \\ \tilde{V}_i \in \frac{1}{n}\mathbb{Z}, & V_i - \frac{1}{n} < \tilde{V}_i \leq V_i, \end{cases} \qquad \text{for } 1 \leq i \leq k-1.$$

Remark that we still have $(\tilde{U}_0, \tilde{V}_0) = (0,0)$ and $(\tilde{U}_k, \tilde{V}_k) = (1,1)$.

Let $X_i = (U_i - U_{i-1}, V_i - V_{i-1})$ and let $\tilde{X}_i = (\tilde{U}_i - \tilde{U}_{i-1}, \tilde{V}_i - \tilde{V}_{i-1})$ be the discrete approximation of X_i for $1 \leq i \leq k$. Conditional on the event that the slopes of $(X_1, \ldots, X_k)$ and $(\tilde{X}_1, \ldots, \tilde{X}_k)$ are pairwise distinct and ranked in the same order, the Hausdorff distance between the associated convex chains is bounded by $\frac{k}{n}$, which goes asymptotically to 0. Since a direct application of [15]*Theorem 2 shows that the distance between the convex chain associated to X and the parabolic arc converges to 0 in probability as k tends to $+\infty$, we deduce that the Hausdorff distance between the convex chain associated to $\tilde{X}$ and the parabolic arc also converges in probability to 0 on this event. As in the proof of Theorem 3.4, the joint density of (X_i, X_j) is dominated by the density of a couple of independent vectors whose coordinates are independent exponential variables with parameter k. These vectors being of order of magnitude $\frac{1}{k}$, the order of the slopes of (X_i, X_j) and $(\tilde{X}_i, \tilde{X}_j)$ may be reversed only if the angle between X_i and X_j is smaller than $\frac{ck}{n}$ for some $c > 0$, which happens with probability of order $\frac{k}{n}$. Henceforth, the probability that there exists $i < j$ for which the slopes of (X_i, X_j) and $(\tilde{X}_i, \tilde{X}_j)$ are ranked in opposite is bounded, up to a constant, by $\binom{k}{2}\frac{k}{n}$. Therefore, the Hausdorff distance between the convex chain associated to $\tilde{X}$ and the parabolic arc also converges to 0 in probability if $k = o(n^{1/3})$.

The final step is to compare the distribution of the increasing reordering of $(\tilde{X}_1, \ldots, \tilde{X}_k)$ with the uniform distribution on $\Pi(n;k)$. As a consequence of Theorem 3.4, the probability that a uniformly random element of $\Pi(n;k)$ is strictly North-East tends to 1. The key points, which follows from Valtr's observation, is that the uniform distribution on strictly North-East convex chains with k edges coincides with the distribution of the chain obtained by reordering the vectors $(\tilde{X}_1, \ldots, \tilde{X}_k)$, conditional on the event that these vectors are pairwise linearly independent and strictly North-East. Since we showed in the previous paragraph that all the angles between two vectors of $(\tilde{X}_1, \ldots, \tilde{X}_k)$ are at least $\frac{ck}{n}$ with probability $1 - O(\frac{k^3}{n})$, the linear independence condition occurs with probability tending to 1. On the other hand, $(\tilde{X}_1, \ldots, \tilde{X}_k)$ are strictly North-East with probability $1 - O(\frac{k^2}{n})$. Therefore, the event we conditioned on has a probability tending to 1, which proves that the total variation distance between the two distributions tends to 0. □

3.6 Back to Jarník's problem

In [33], Jarníkgives an asymptotic formula of the maximum possible number of vertices of a convex lattice polygonal line having a *total Euclidean length* smaller than n, and whose segments make an angle with the x-axis between 0 and $\frac{\pi}{4}$. What he finds is

$\frac{3}{2}\frac{n^{2/3}}{(2\pi)^{1/3}}$. If, in order to be closer to our setting, we ask the segments to make an angle with the x-axis between 0 and $\frac{\pi}{2}$, Jarník's formula is changed into $\frac{3}{2}\frac{n^{2/3}}{\pi^{1/3}}$ (which is twice the above result for $\frac{n}{2}$).

In this section, we want to present a detailed combinatorial analysis of this set of lines, which leads to Jarník's result as well as to the asymptotic of the *typical* number of vertices of such lines. It is the analog of Bárány, Sinaïand Vershik's result when the constraint concerns the total length.

Let us first describe Jarník's argument, which is a good application of the correspondence described in Section 3.2. It says the following: the function ω realizing the maximum can be taken among the functions taking their values in $\{0,1\}$. Indeed, by changing the non-zero values of a function ν into 1, one can obtain a chain with the same number of vertices, but with a shorter length. Now, if the number of vertices k is given, the convex chain having minimal length, will be defined by the function ω which associates 1 to the k points of $\mathbb{X}$ which are the closest to the origin. Since the set X has an asymptotic density $\frac{6}{\pi^2}$, when N is big, this set of points is asymptotically equivalent to the intersection of X with the disc of center O having radius R satisfying $\frac{6}{\pi^2}\cdot\frac{\pi R^2}{4} = N$ i.e. $R = (\frac{2\pi}{3}N)^{1/2}$. The total length of the line is equivalent to $L = \int_0^R r\times\frac{6}{\pi^2}\frac{\pi}{2}r\,dr = \frac{R^3}{\pi} = \frac{(\frac{2\pi}{3}N)^{3/2}}{\pi}$. This yields precisely $N = \frac{3}{2}\frac{L^{2/3}}{\pi^{1/3}} \simeq 1.02\,L^{2/3}$.

In order to get finer results, we introduce the probability distribution on the space Ω proportional to

$$\exp\left(-\beta\sum_{x\in\mathbb{X}}\omega(x)\sqrt{|x_1|^2+|x_2|^2}\right)\lambda^{\sum_{x\in\mathbb{X}}1_{\{\omega(x)>0\}}}$$

which depends on two parameters β,λ. In this set-up, the partition function turns out to be

$$Z = \prod_{x\in\mathbb{X}}\frac{1-(1-\lambda)e^{-\beta\sqrt{|x_1|^2+|x_2|^2}}}{1-e^{-\beta\sqrt{|x_1|^2+|x_2|^2}}}.$$

The Mellin transform representation for $\log Z$ now involves

$$\frac{\Gamma(s)(\mathrm{Li}_{s+1}(1-\lambda)-\zeta(s+1))}{\zeta(s)}\sum_{x_1,x_2\geq 1}(|x_1|^2+|x_2|^2)^{-s/2}, \qquad \Re\mathfrak{e}(s)>2.$$

The factors $\zeta(s)^{-1}$ and $\mathrm{Li}_{s+1}(1-\lambda)-\zeta(s+1)$, which correspond respectively to the coprimality condition on the lattice and to the penalization of vertices, are still present. The main difference relies in the replacement of the Barnes zeta function by the Epstein zeta function which comes from the penalization by length in the model. With the help of the residue analysis of this Mellin transform and a local limit theorem, we obtain:

Theorem 3.6. *Let $p_J(n;k)$ denote the number of convex chains on $\mathbb{Z}_+^2$ issuing from $(0,0)$ with k vertices and length between n and $n+1$. As n tends to $+\infty$,*

$$\text{if}\qquad \frac{k}{n^{2/3}}\longrightarrow\frac{\pi^{1/3}}{2}\mathbf{c}(\lambda),\qquad\text{then}\qquad \frac{1}{n^{2/3}}\log p_J(n;k)\longrightarrow\frac{\pi^{1/3}}{2}\mathbf{e}(\lambda),$$

where $\mathbf{e}$ *and* $\mathbf{c}$ *are the functions introduced in Theorem 3.1. Moreover, the Hausdorff distance between a random element of this set normalized by* $\frac{1}{n}$, *and the arc of circle* $\{(x,y)\in[0,1]^2 : x^2+(y-1)^2=1\}$ *converges to* 0 *in probability.*

From this result, we deduce that the typical number of vertices of such a chain which is achieved for $\lambda = 1$ is asymptotically equal to

$$\left(\frac{3}{4\pi\zeta(3)^2}\right)^{1/3} n^{2/3}.$$

Similarly, the total number of convex chains having length between n and $n+1$ is asymptotically equal to

$$\exp\left(\frac{3^{4/3}\zeta(3)^{1/3}}{(4\pi)^{1/3}} n^{2/3}(1+o(1))\right).$$

In addition, we can derive Jarník's result in the lines of Remark 3.3.

3.7 Mixing constraints and finding new limit shapes

In this section we introduce a family of lattice convex chain models which achieves a continuous interpolation of limit shapes between the diagonal of the square and the South-East corner sides of the square, passing through the arc of circle and the arc of parabola. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote respectively the Taxicab norm and the Euclidean norm on $\mathbb{R}^2$. Recall that for all $x\in\mathbb{R}^2$,

$$\|x\|_1 = |x_1|+|x_2| \geq \|x\|_2 = \sqrt{|x_1|^2+|x_2|^2} \geq \frac{1}{\sqrt{2}}\|x\|_1.$$

The Gibbs distribution we consider on the space Ω involves both these norms in order to take into account both the extreme point of the chain and its length:

$$\frac{1}{Z}\exp\left(-\beta\sum_{x\in\mathbb{X}}\omega(x)(\|x\|_1+\lambda\sqrt{2}\|x\|_2)\right), \quad Z=\prod_{x\in\mathbb{X}}\left(1-e^{-\beta(\|x\|_1+\lambda\sqrt{2}\|x\|_2)}\right).$$

This infinite product is convergent if $\beta>0$ and $\lambda>-\frac{1}{\sqrt{2}}$ or if $\beta<0$ and $\lambda<-1$. In both cases, the Mellin transform representation of $\log Z$ involves

$$\frac{\Gamma(s)\zeta(s+1)}{\zeta(s)}\sum_{x_1,x_2\geq 1}(\|x\|_1+\lambda\sqrt{2}\|x\|_2)^{-s}, \qquad \Re(s)>2.$$

As usual, the leading term of the expansion of $\log Z$ when $\beta\to 0$ is obtained by computing the residue of this function at $s = 2$. It turns out to be

$$\frac{\zeta(3)}{2\zeta(2)} \int_{-\pi/4}^{\pi/4} \frac{d\theta}{(\lambda + \cos(\theta))^2}.$$

An application of the residue theorem shows that the expected length of the curve is asymptotically equivalent to

$$\frac{1}{\beta^3} \frac{\zeta(3)}{\sqrt{2}\zeta(2)} \int_{-\pi/4}^{\pi/4} \frac{d\theta}{(\lambda + \cos(\theta))^3}$$

and that the coordinates of the ending point have asymptotic expected value

$$\frac{1}{\beta^3} \frac{\zeta(3)}{2\zeta(2)} \int_{-\pi/4}^{\pi/4} \frac{\cos(\theta) d\theta}{(\lambda + \cos(\theta))^3}.$$

As in previous sections, a local limit theorem gives a correspondence between this Gibbs measure and the uniform distribution on a specific set of convex chains, namely the convex chains with endpoint (n,n) and total length belonging to $[L \cdot n, L \cdot n + 1]$ for some $L \in]\sqrt{2}, 2[$ which is a function of λ,

$$L(\lambda) = \frac{\sqrt{2} \int_0^{\frac{\pi}{4}} \frac{1}{(\lambda + \cos u)^3} du \int_0^{\frac{\pi}{4}} \frac{\cos u}{(\lambda + \cos u)^3} du}{.}$$

By computations analogous to Section 3.4, one can show that the uniform distribution of chains with length between $L(\lambda) \cdot n$ and $L(\lambda) \cdot n + 1$ concentrates around the curve described by the parametrization

$$x_\lambda(\phi) = \sqrt{2} \frac{\int_0^{\phi} \frac{\cos u}{(\lambda + \cos(u - \frac{\pi}{4}))^3} du}{\int_{-\pi/4}^{\pi/4} \frac{\cos u}{(\lambda + \cos u)^3} du}, \quad y_\lambda(\phi) = \sqrt{2} \frac{\int_0^{\phi} \frac{\sin u}{(\lambda + \cos(u - \frac{\pi}{4}))^3} du}{\int_{-\pi/4}^{\pi/4} \frac{\cos u}{(\lambda + \cos u)^3} du} \quad (0 \le \phi \le \frac{\pi}{2}).$$

The table provided in Figure 3.2 on the following page resumes the limit shapes that we obtain for some limit values of λ. See also Figure 3.3 for a plot showing the interpolation of those limit shapes.

3.8 An asymptotic equivalent for the number of convex chains

In this section, we push some steps further the asymptotic analysis of Section 3.3. In order to make the expressions simpler and the calculations easier to follow, we specify the model of Section 3.3 with $\lambda = 1$ and $\beta_1 = \beta_2 = \beta$. This corresponds to Sinaĭ's original model. In this case, the formula of Lemma 3.3 for the logarithmic partition function can

λ	$-\infty$	-1	$-\frac{1}{\sqrt{2}}$	0	$+\infty$
Limit shape	circle	diagonal	square	parabola	circle
Length $L(\lambda)$	$\frac{\pi}{2}$	$\sqrt{2}$	2	$1+\frac{\ln(1+\sqrt{2})}{\sqrt{2}}$	$\frac{\pi}{2}$

Figure 3.2 – Critical and special values in the spectrum of limit shapes for the model of lattice convex chains with mixed constraints.

be written in terms of univariate zeta functions only,

$$\log Z(\beta) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\zeta(s+1)(\zeta(s-1)+\zeta(s))}{\zeta(s)\beta^s}\,ds$$

where the integral is convergent for all $\beta > 0$ and $c > 2$. Recall indeed that $|\Gamma(s)|$ has an exponential decay as $|\mathfrak{Im}(s)| \to +\infty$ and $|\zeta(s)|$ is at most polynomially increasing, uniformly in any vertical strip. Similarly, an upper bound of $1/\zeta(s)$ is easily obtained outside the critical strip by using its Dirichlet series expansion and the functional equation of ζ.

As in Lemma 3.4, one can derive an asymptotic series expansion of $\log Z(\beta)$ from this formula by using the residue theorem. The singularity of the integrand with largest real part lies at $s = 2$. It is a pole of order 1 and it comes with a residue $\zeta(3)/\zeta(2)\beta^{-2}$. The next singularity at, $s = 1$, is actually removable thanks to the presence of $\zeta(s)$ in the denominator. Next, one enters the critical strip $0 < \mathfrak{Re}(s) < 1$ where the integrand has a pole at each zero of $\zeta(s)$. At $s = 0$, both $\Gamma(s)$ and $\zeta(s+1)$ have a pole. A straightforward computation yields

$$\operatorname{Res}_{s=0}\left(\frac{\Gamma(s)\zeta(s+1)(\zeta(s-1)+\zeta(s))}{\zeta(s)\beta^s}\right) = \frac{7}{6}\log\frac{1}{\beta} - 2\zeta'(-1) - \frac{1}{6}\log(2\pi).$$

Finally, the residue of the pole at $s = -1$ is $\frac{\beta}{2}$, the singularities of $\Gamma(s)$ at $s = -2k-1$ for $k \in \mathbb{N}$ are removed by the presence of $\zeta(s+1)$, and there are poles of order 2 at $s = -2k$ for $k \in \mathbb{N}$.

Let γ be the contour defined as the union of the following oriented paths: on the right side the curve $\gamma_{\text{right}}(t) = 1 - \frac{A}{\log(2+|t|)} + it$ for t going from $-\infty$ to $+\infty$ and on the left side the union γ_{left} of the vertical half-lines with respective origin $-\frac{1}{2} \pm i$ joined by the semicircle of center $-\frac{1}{2}$ and radius 1 going through $\frac{1}{2}$ traveled from $+i\infty$ to $-i\infty$. The value of $A > 0$ is chosen so that the contour γ circles around all non trivial zeros of the zeta function (the curve γ_{right} corresponds actually to the boundary of a classical zero-free region, see [50]*Theorem 3.8). Therefore, the corresponding contribution in

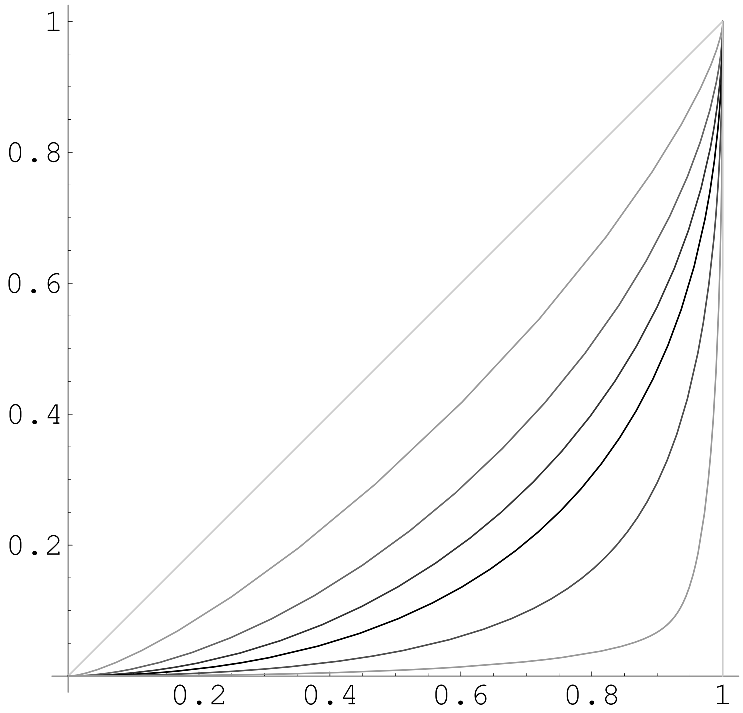

Figure 3.3 – Limit shapes of different Euclidean lengths. Successively: $\sqrt{2}$ (diagonal); 1.454, 1.516, $\frac{\pi}{2}$ (circle), $1+\frac{\ln(1+\sqrt{2})}{\sqrt{2}}$ (parabola), 1.716, 1.861 and 2 (square).

$\log Z(\beta)$ will be given by

$$H(\beta) = \frac{1}{2i\pi}\int_{\gamma} \frac{\Gamma(s)\zeta(s+1)(\zeta(s-1)+\zeta(s))}{\zeta(s)\beta^s} ds. \tag{3.14}$$

This function is well defined since the integral is convergent. Moreover, this $H(\beta)$ is a $o(\beta^{-1})$ as $\beta \to 0$. Indeed, this is easily seen to be true for the integral along γ_{left} since this integral is at most of order $\beta^{-1/2}$. The integral along γ_{right} is dealt with Lebesgue's dominated convergence theorem, the domination being obtained using [50]*Theorem 3.11. Similarly, the derivatives $H'(\beta)$ and $H''(\beta)$ are respectively $o(\beta^{-2})$ and $o(\beta^{-3})$.

In order to deal with a contour crossing the critical strip, we will need the following result of Valiron [50]*Theorem 9.7: there exists $\alpha > 0$ and a sequence (T_k) such that for all $k \in \mathbb{N}$, $k < T_k < k+1$ and $|\zeta(s)| > |\mathfrak{Im}(s)|^{-\alpha}$ uniformly for all s such that $|\mathfrak{Im}(s)| = T_k$ and $-1 \leq \mathfrak{Re}(s) \leq 2$. Therefore, if one applies the residue theorem with the rectangle $[3 \mp iT_k, -\frac{1}{2} \pm iT_k]$ and let k tends to $+\infty$, the contributions of the horizontal segments

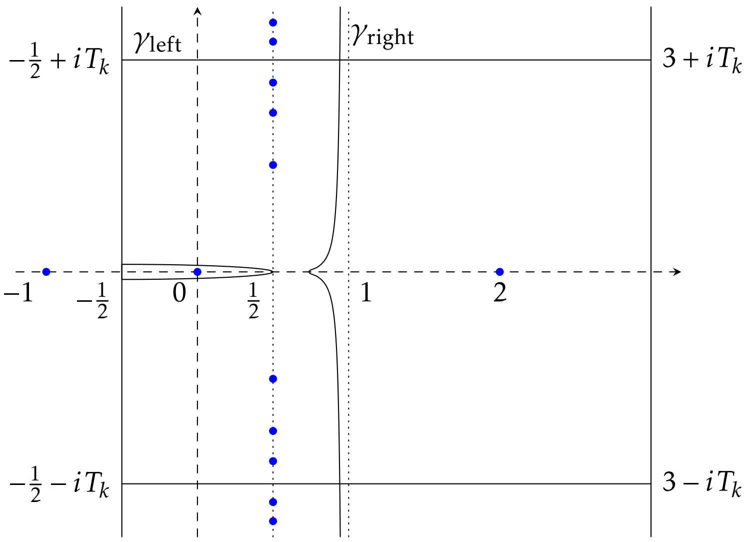

Figure 3.4 – Localization of poles (in blue) and contours of integration with which the residue theorem is applied to get the asymptotic expansion of $\log Z(\beta)$ as β tends to 0.

vanish and one gets

$$\log Z(\beta) = \frac{\zeta(3)}{\zeta(2)}\frac{1}{\beta^2} + H(\beta) + \frac{7}{6}\log\frac{1}{\beta} + C + I(\beta) \tag{3.15}$$

with $C = -2\zeta'(-1) - \frac{1}{6}\log(2\pi)$ and where the error term

$$I(\beta) = \frac{1}{2i\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(s)\zeta(s+1)(\zeta(s-1)+\zeta(s))}{\zeta(s)\beta^s}ds$$

is at most of order $O(\beta^{1/2})$. Moreover, we have

$$H(\beta) = \lim_{k\to\infty}\sum_{|\mathfrak{Im}(\rho)|<T_k} \frac{\Gamma(\rho)\zeta(\rho+1)\zeta(\rho-1)}{\zeta'(\rho)\beta^\rho} \tag{3.16}$$

where ρ runs through the zeros of ζ with $0 < \mathfrak{Re}(\rho) < 1$. Here we have assumed for notational simplicity that these zeros have multiplicity 1 but this is not restrictive. Note that term-by-term differentiation with respect to β in the formula (3.15) can be justified by differentiation in the convergent integral representation of $\log Z(\beta)$, $H(\beta)$ and $I(\beta)$. Remark that in the previous arguments, one could push the left-side of the rectangular contour of integration as far as needed to the left in order to obtain complete asymptotic series expansions.

Now, let us define the calibrated parameter β, as in Section 3.3.3, by the equation

$\frac{d}{d\beta}\log Z(\beta) = -2n$. Using the previous estimates for H' and H'', we obtain

$$\frac{1}{\beta^3} = \frac{n}{\kappa} + \frac{1}{2\kappa}H'\left(\left(\frac{\kappa}{n}\right)^{1/3}\right) - \frac{7}{12\kappa}\left(\frac{n}{\kappa}\right)^{1/3} + o(n^{1/3}) \tag{3.17}$$

with $\kappa = \zeta(3)/\zeta(2)$. For this calibrated parameter, a local limit theorem, which is a simplified version of Theorem 3.2, gives

$$\frac{1}{Z(\beta)}e^{-2n\beta}p(n) = \mathbb{P}_\beta(X = n) \sim \frac{\kappa^{1/3}}{2\pi\sqrt{3}n^{4/3}} \tag{3.18}$$

where the right-hand term corresponds, up to 2π, to the inverse of the standard deviation of X under $\mathbb{P}_\beta$. This is nothing but the square root of the determinant of the Hessian matrix of $\log Z(\beta)$ at β which is asymptotically equal to

$$\begin{vmatrix} \frac{2\kappa}{\beta^4} & \frac{\kappa}{\beta^4} \\ \frac{\kappa}{\beta^4} & \frac{2\kappa}{\beta^4} \end{vmatrix} = \frac{3\kappa^2}{\beta^8}.$$

Gathering (3.15), (3.17), (3.18), we are able to state

Theorem 3.7. *The number $p(n)$ of lattice convex chain in $[0,n]^2$ from $(0,0)$ to (n,n) satisfies*

$$p(n) \underset{n\to\infty}{\sim} \frac{e^{-2\zeta'(-1)}}{(2\pi)^{7/6}\sqrt{3}\kappa^{1/18}n^{17/18}} \exp\left[3\kappa^{1/3}n^{2/3} + H\left((\kappa/n)^{1/3}\right)\right], \tag{3.19}$$

with $\kappa = \zeta(3)/\zeta(2)$ and where H can be defined either by (3.14) *or by* (3.16).

As we observed above, the fact that the additional term $H((\kappa/n)^{1/3})$ is at most of order $o(n^{1/3})$ follows from the existence of large zero-free regions of the Riemann zeta function in the critical strip. This estimation can be considerably improved if one assumes the truth of Riemann's Hypothesis, namely that all zeros of the zeta function in the critical strip lie actually on the critical line $\mathfrak{Re}(s) = \frac{1}{2}$. Under this assumption, the oscillating quantity $H((\kappa/n)^{1/3})$ is at most of order $o(n^{1/6+\varepsilon})$ for all $\varepsilon > 0$.

In view of numerical computations, one can truncate the series defining H in (3.16) at the two conjugate zeros of zeta on the critical line with smallest imaginary part, which is approximately 14.1347. Indeed, the next zeros have an imaginary part around 21.0220 and the exponential decrease of the Γ function makes this next term around 10^4 times smaller than the first one. Therefore, with a relative precision of 10^{-4},

$$H(\beta) \approx \frac{1}{\sqrt{\beta}}\left[6.0240\cdot 10^{-11}\cos(14.1347\log\beta) + 9.5848\cdot 10^{-10}\sin(14.1347\log\beta)\right]$$

for usual values of $\beta = (\kappa/n)^{1/3}$. In addition, one can see from this approximation that for a large range of values of n, the importance of the oscillatory term in the exponential part of (3.19) is actually extremely small in comparison with the polynomial pre-factor.

Chapter 4

Limit shapes of zonotopes and integer partitions

The work presented in this chapter has been realized in collaboration with Imre Bárány. We explore the link between lattice zonotopes and strict integer partitions and we use this link to estimate the number of zonotopes, as well as to describe limit shapes for random zonotopes under the uniform distribution.

4.1 General overview

The work presented in this chapter takes place in a general study of asymptotic properties of large convex lattice polytopes and integer partitions in the affine space $\mathbb{R}^d$, which is exposed with more details by Vershik [52] and Bárány [11]. In 1994, Vershik [53], Bárány [8] and Sinaĭ [44] proved the existence of a limit shape for the set of all convex lattice polygons lying in the square $[-n,n]^2$ endowed with the uniform distribution. Their approaches all rely on a deep link between lattice convex polygons on the first hand, and integer partitions on the other hand. There is indeed a one-to-one correspondence ([44, Lemma 1]) between the set of convex lattice polygonal lines from $\mathbf{0} = (0,0)$ to $\mathbf{n} = (n,n)$ with vertices in $([0,n] \cap \mathbb{Z})^2$ and the set of strict partitions of the vector $\mathbf{n}$. Let us recall that a strict partition of a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$ is a partition in which each part is a two-dimensional vector whose components are coprime nonnegative integers.

When dealing with a higher dimension $d \geq 3$, there is no such simple connection between convex lattice polytopes lying in $[-n,n]^d$ and integer partitions. But there still is a correspondence between strict d-partite integer partitions and lattice *zonotopes*, which are convex polytopes obtained by taking the convex hull of the family vectors

$$\sum_{i=1}^{k} \varepsilon_i \mathbf{v}^i, \qquad (\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k$$

for a given finite family of vectors $\mathbf{v}^1,\ldots,\mathbf{v}^k$ in $\mathbb{Z}^d$. When each one of the vectors $\mathbf{v}^1,\ldots,\mathbf{v}^k$ is located on the same side of a linear hyperplane H, the vector $\mathbf{v}^1+\cdots+\mathbf{v}^k$ is the extremal point of the zonotope with maximal distance from H, which we call the *ending point*. Even in this case, the total number of zonotopes with given ending point is usually infinite. But if we add the restriction that the vectors are to be taken in a salient convex cone C, there is only a finite number $p_C(\mathbf{n})$ of zonotopes with given ending point $\mathbf{n} \in C$. The case of the nonnegative orthant $C = [0,\infty)^d$ is the most natural, but our approach is relevant for nearly any cone.

We focus in this paper on the asymptotic behaviour of the uniform distribution on the set of lattice zonotopes with vectors in C and with ending point $\mathbf{n}$, when $\|\mathbf{n}\|$ goes to infinity. Our first result (Theorem 4.1 on page 86) is a logarithmic estimate of $p_C(\mathbf{n})$. The main result of the paper (Theorem 4.2 on page 93)is the existence of a deterministic limit shape for random zonotopes with respect to the uniform distribution. Both the logarithmic estimate and the limit shape are easily described in terms of the Laplace transform of the cone, that we define in Section 4.3.

The proof of these results is based on a generalization of Sinaĭ's probabilistic method [44] for convex chains. It introduces a probability measure on the set of lattice zonotopes which is akin to the *grand canonical ensemble* in statistical mechanics. This model is presented in Section 4.4.2. The bridge between this distribution and the uniform distribution depends on a generic *local central limit theorem* which was presented and applied to bipartite integer partitions in [19].

Another direction of study about limit shapes of zonotopes has been considered by Davydov and Vershik [21]. It covers the case where the generating vectors $\mathbf{v}_1,\ldots,\mathbf{v}_k$ of the zonotope are independent random vectors with identical distribution. Such random zonotopes are intimately related to convex rearrangements of random walks. Davydov and Vershik interpret these zonotopes as random walks in the Banach space of convex subsets of $\mathbb{R}^d$, where the additive law is the Minkowski addition. They prove a Strong Law of Large Numbers and Central Limit Theorems for these random walks.

4.2 From lattice zonotopes to strict integer partitions

For every finite unordered family of lattice vectors $\mathbf{v}^1,\ldots,\mathbf{v}^k \in \mathbb{Z}^d$ let the zonotope generated by this family be the convex hull

$$\mathcal{Z}(\mathbf{v}^1,\ldots,\mathbf{v}^k) = \mathrm{Conv}\left\{\sum_{i=1}^k \varepsilon_i \mathbf{v}^i;\ (\varepsilon_1,\ldots,\varepsilon_k) \in \{0,1\}^k\right\}.$$

A zonotope is a convex polytope. Its *support function* $h_{\mathcal{Z}}(\cdot) = \sup_{\mathbf{v}\in\mathbb{Z}} \langle \cdot, \mathbf{v}\rangle$ admits a particularly simple expression, which can be used to give an alternative description of the zonotope:

$$h_{\mathcal{Z}}(\mathbf{u}) = \sum_{i=1}^k \langle \mathbf{u}, \mathbf{v}_i\rangle \mathbf{1}_{\{\langle \mathbf{u},\mathbf{v}^i\rangle \geq 0\}}, \qquad \mathbf{u} \in \mathbb{R}^d.$$

Since this formulation only depends on the directions of the vectors, it makes it clear that every lattice zonotope is generated by a family of *primitive vectors* $\mathbf{w}^1,\ldots,\mathbf{w}^\ell$ such that $\gcd(w_1^j,\ldots,w_d^j) = 1$ for $1 \le j \le \ell$. The family of primitive vectors is obtained by the following procedure: replace each vector $\mathbf{v}^i$ by $D^i = \gcd(\mathbf{v}_1^i,\ldots,\mathbf{v}_d^i)$ occurrences of the primitive vector $\frac{1}{D^i}\mathbf{v}^i$. Notice that $\ell = \sum_{i=1}^k \gcd(\mathbf{v}_1^i,\ldots,\mathbf{v}_d^i)$ and, more importantly, notice that the total sum is preserved,

$$\sum_{i=1}^{k} \mathbf{v}^i = \sum_{j=1}^{\ell} \mathbf{w}^j.$$

In addition, the generating family made of primitive vectors is *unique*, up to the order of the terms. Let us prove this fact. Let $\mathbf{v}^1,\ldots,\mathbf{v}^k$ and $\mathbf{w}^1,\ldots,\mathbf{w}^\ell$ be family of primitive lattice vectors that generate the same zonotope $\mathcal{Z}$. Let m be the multiplicity of $\mathbf{v}^1$. Let $\mathbf{u} \in \mathbb{R}^d \setminus \{0\}$ be such that $\left\langle \mathbf{u}, \mathbf{v}^1 \right\rangle = 0$. From the expression of the support function $h_{\mathcal{Z}}$ in terms of the vectors $\mathbf{v}^1,\ldots,\mathbf{v}^k$, one see that we can find an extremal point $\mathbf{a}$ of $\mathcal{Z}$ such that $\mathbf{a} + m\mathbf{v}^1$ is also an extremal point of $\mathcal{Z}$ and $h_{\mathcal{Z}}(\mathbf{u}) = \langle \mathbf{u}, \mathbf{a} \rangle$. Now, the expression of $h_{\mathcal{Z}}$ in terms of the vectors $\mathbf{w}^1,\ldots,\mathbf{w}^\ell$ shows that $m\mathbf{v}^1$ can be written as a sum of vectors $\mathbf{w}^j$ such that $\left\langle \mathbf{u}, \mathbf{w}^j \right\rangle = 0$. Since this must be true for every vector $\mathbf{u}$ orthogonal to $\mathbf{v}^1$, we deduce that these vectors $\mathbf{w}^j$ are parallel to $\mathbf{v}^1$. Since the vectors are primitive, it implies that there exists j such that $\mathbf{w}^j = \mathbf{v}^i$, and the multiplicities of this common vector in both families are equal.

Let $\mathbb{Z}_*^d = \{\mathbf{x} \in \mathbb{Z}^d : \gcd(x_1,\ldots,x_d) = 1\}$ be the set of primitive vectors. For obvious reasons, this set is also often refered to as the set of lattice points visible from the origin. Notice that $\mathbb{Z}_*^d$ does not contain the zero vector. We have just defined a one-to-one correspondence between lattice zonotopes generated by a family of vectors $\mathbf{v}^1,\ldots,\mathbf{v}^k$ such that $\sum_{i=1}^k \mathbf{v}^i = \mathbf{n}$ on the first hand, and families of primitive vectors $\mathbf{w}^1,\ldots,\mathbf{w}^\ell$ such that

$$\mathbf{w}^1 + \mathbf{w}^2 + \ldots \mathbf{w}^\ell = \mathbf{n}$$

on the other hand.

Such a family is called a *strict integer partition* of the vector $\mathbf{n} \in \mathbb{Z}^d$. It can be alternatively described by the family $(\omega(\mathbf{x}))_{\mathbf{x}\in\mathbb{Z}_*^d}$ of multiplicities $\omega(\mathbf{x}) = \left|\left\{j \in \{1,\ldots,\ell\} : \mathbf{w}^j = \mathbf{x}\right\}\right|$ of the available parts $\mathbf{x} \in \mathbb{Z}_*^d$. Notice that for any partition, there is only a finite number of vectors $\mathbf{x} \in \mathbb{Z}_*^d$ such that $\omega(\mathbf{x}) \neq 0$. Therefore, picking a strict partition is actually equivalent to picking a function $\omega : \mathbb{Z}_*^d \to \mathbb{Z}_+$ with finite support. For any such function ω, let us define

$$\mathbf{X}(\omega) := \sum_{\mathbf{x}\in\mathbb{Z}_*^d} \omega(\mathbf{x})\mathbf{x}.$$

With this notation, the fact that ω describes a partition of $\mathbf{n}$ corresponds to the condition $\mathbf{X}(\omega) = \mathbf{n}$.

The following lemma summarizes what we have proven here. The second part of the statement is obtained by particularizing the previous discussion to lattice zonotopes with generating vectors in a positive convex cone C and to strict partitions with parts in

C.

Lemma 4.1. *Let $\mathbf{n} \in \mathbb{Z}^d$. The set of lattice zonotopes $\mathcal{Z}(\mathbf{v}^1, \dots, \mathbf{v}^k)$ such that $\mathbf{v}^1 + \cdots + \mathbf{v}^k = \mathbf{n}$ is in one two one correspondence with the set of strict integer partitions of $\mathbf{n}$.*

Let C be a positive convex cone in $\mathbb{R}^d$. Let $\mathbf{n} \in \mathbb{Z}^d$. The set of lattice zonotopes $\mathcal{Z}(\mathbf{v}^1, \dots, \mathbf{v}^k) \subset C$ such that $\mathbf{v}^1 + \cdots + \mathbf{v}^k = \mathbf{n}$ is in one two one correspondence with the set of strict integer partitions of $\mathbf{n}$ with parts in C.

4.3 Directed cones and their Laplace transform

Let C be a closed convex cone in $\mathbb{R}^d$ with non-empty interior. We will furthermore assume that $C \setminus \{0\}$ is contained in an open half-space, or equivalently that the set

$$\mathcal{D}(C) = \{\mathbf{u} \in \mathbb{R}^d : \forall \mathbf{x} \in C \setminus \{0\}, \langle \mathbf{u}, \mathbf{x} \rangle > 0\}$$

is non-empty. If $\mathbf{u}$ is an element of $\mathcal{D}(C)$, we say that $(C, \mathbf{u})$ is a *directed cone*. The *direction* of a directed cone $(C, \mathbf{u})$ is the vector $\mathbf{u}$.

The next proposition allows us to define the *Laplace transform* of C, which is a powerful tool for understanding the mass distribution of the cone.

Proposition 4.2. *The dual cone $\mathcal{D}(C)$ is an open convex cone of $\mathbb{R}^d$. For all $\mathbf{u} \in \mathcal{D}(C)$, the non-negative integral*

$$\Lambda_C(\mathbf{u}) = \int_C e^{-\langle \mathbf{u}, \mathbf{x} \rangle} d\mathbf{x}$$

is convergent. The function Λ_C thus defined is a smooth function in $\mathcal{D}(C)$.

Proof. It is clear that $\mathcal{D}(C)$ is a convex cone because it is an intersection of half-spaces. Notice that $\mathbf{u} \in \mathcal{D}(C)$ if and only if $\langle \mathbf{u}, \mathbf{x} \rangle > 0$ for all unit vector $\mathbf{x} \in C$, by homogeneity of the norm. Since the intersection of C with the unit sphere $\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$ is both closed and bounded in $\mathbb{R}^d$, it is compact and

$$f(\mathbf{u}) = \min_{\mathbf{x} \in \mathbb{S} \cap C} \langle \mathbf{u}, \mathbf{x} \rangle$$

is well defined for $u \in \mathbb{R}^d$. Moreover the function f is 1-Lipschitz, hence it is continuous on $\mathbb{R}^d$. We conclude that $\mathcal{D}(C)$ is open because it is the inverse image of $(0, +\infty)$ by f.

Every $\mathbf{u} \in \mathcal{D}(C)$ satisfies $f(\mathbf{u}) > 0$ by compactness of $\mathbb{S} \cap C$, and for all $\mathbf{x} \in C$, the inequality

$$\langle \mathbf{u}, \mathbf{x} \rangle \geq f(\mathbf{u}) \|\mathbf{x}\|$$

holds by definition of f. Hence,

$$\langle \mathbf{u}, \mathbf{x} \rangle \geq f(\mathbf{u}) \sqrt{|x_1|^2 + \cdots |x_d|^2} \geq \frac{f(\mathbf{u})}{\sqrt{2}} (|x_1| + \cdots + |x_d|).$$

We can use this inequality to bound $\Lambda_C(\mathbf{u})$ for $\mathbf{u} \in \mathcal{D}(C)$, namely

$$\Lambda_C(\mathbf{u}) \leq \int_C e^{-\langle \mathbf{u}, x\rangle} d\mathbf{x} \leq \left(\int_{\mathbb{R}} e^{-\frac{f(\mathbf{u})}{\sqrt{2}}|x|} dx \right)^d = \left(\frac{2\sqrt{2}}{f(\mathbf{u})} \right)^d < \infty.$$

The smoothness of Λ_C is obtained similarly by differentiating under the integral. □

The Laplace transform satisfies the following useful homogeneity property.

Proposition 4.3. *If T is an invertible linear transformation of $\mathbb{R}^d$ with adjoint T^*, then $T(C)$ is a closed convex cone with non-empty interior. Its open dual cone is $\mathcal{D}(T(C)) = (T^*)^{-1}(\mathcal{D}(C))$. For all $\mathbf{u} \in \mathcal{D}(T(C))$,*

$$\Lambda_{T(C)}(\mathbf{u}) = |\det T| \times \Lambda_C(T^*\mathbf{u}).$$

Proof. This a merely a linear change of variables in the integral. □

In the special case of the homothetic transformation $T = t\,\mathrm{Id}$ with ratio $t > 0$, the cone C is preserved and the relation of homogeneity is simply

$$\Lambda_C(t\mathbf{u}) = \frac{1}{t^d}\Lambda_C(\mathbf{u}).$$

In accordance with these observations, the Laplace transform can be interpreted as an elementary measure of volume of the cone C (the volume of C itself is of course infinite). The next proposition will make this geometric interpretation more precise. If $(C, \mathbf{u})$ is a directed cone, we define the *truncated cone* of C in the direction of $\mathbf{u}$ as

$$C_{\mathbf{u}} = \{\mathbf{x} \in C : \langle \mathbf{u}, \mathbf{x}\rangle \leq 1\}.$$

We also define the *exponential distribution* on the directed cone $(C, \mathbf{u})$. This is the probability distribution which is absolutely continuous with respect to the Lebesgue measure on C with density

$$f_{(C,\mathbf{u})} : \mathbf{x} \mapsto \frac{e^{-\langle \mathbf{u}, \mathbf{x}\rangle}}{\Lambda_C(\mathbf{u})}.$$

The following proposition relates $\Lambda_C(u)$, the exponential distribution, and the uniform distribution on $(C, \mathbf{u})$, providing some geometric and probabilistic insight on these notions.

Proposition 4.4. *Let $(C, \mathbf{u})$ be a directed cone. The volume of the truncated cone $C_{\mathbf{u}}$ is*

$$|C_{\mathbf{u}}| = \frac{1}{d!}\Lambda_C(\mathbf{u}).$$

Moreover, if ξ is a random vector following the exponential distribution on $(C, \mathbf{u})$, then the expected value of ξ is

$$-\frac{\nabla \Lambda_C(\mathbf{u})}{\Lambda_C(\mathbf{u})} = (d+1) \int_{C_{\mathbf{u}}} \frac{\mathbf{x}\, d\mathbf{x}}{|C_{\mathbf{u}}|}$$

and the second order moments of ξ are given by

$$\frac{1}{\Lambda_C(\mathbf{u})}\frac{\partial^2\Lambda_C(\mathbf{u})}{\partial u_i \partial u_j} = (d+1)(d+2)\int_{C_\mathbf{u}} \frac{x_i x_j \, d\mathbf{x}}{|C_\mathbf{u}|}.$$

Proof. Since $e^{-\langle \mathbf{u},\mathbf{x}\rangle} = \int_{\langle \mathbf{u},\mathbf{x}\rangle}^{\infty} e^{-t}\,dt$, we can write

$$\Lambda_C(\mathbf{u}) = \int_C \int_0^\infty e^{-t}\mathbf{1}_{\{\langle \mathbf{u},\mathbf{x}\rangle \le t\}}\,dt\,dx.$$

The Fubini-Tonelli theorem now yields

$$\Lambda_C(\mathbf{u}) = \int_0^\infty t^d e^{-t}\,dt \int_C \mathbf{1}_{\{\langle \mathbf{u},\mathbf{x}\rangle \le 1\}}\,d\mathbf{x} = \Gamma(d+1)|C_\mathbf{u}|.$$

The other formulae are obtained similarly. □

4.3.1 Examples

— Let Orth $= [0,\infty)^d$ denote the non-negative orthant. Its open dual cone is the positive orthant $(0,\infty)^d$ and for all $\mathbf{u} \in (0,\infty)^d$,

$$\Lambda_{\mathrm{Orth}}(\mathbf{u}) = \frac{1}{u_1 u_2 \cdots u_d}.$$

— Let C be the circular non-negative cone of $\mathbb{R}^3$ of equation $x^2+y^2 \le z^2$ with $z \ge 0$. As with the previous example, the open dual cone $\mathcal{D}(C)$ is simply the interior of C. Since the Laplace transform is homogeneous, it is enough to compute $\Lambda_C(u,v,w)$ on the cylinder $u^2+v^2=1$. Using cylindrical coordinates, we obtain for all $t \in \mathbb{R}$ and $w > 1$,

$$\begin{aligned}
\Lambda_C(\cos(t),\sin(t),w) &= \int_0^\infty \int_0^z \int_{-\pi}^{\pi} e^{-\rho\cos(t-\phi)-wz}\rho\,d\phi\,d\rho\,dz \\
&= 2\pi \int_0^\infty \int_0^z e^{-wz} I_0(\rho)\rho\,d\rho\,dz \\
&= \frac{2\pi}{w}\int_0^\infty e^{-w\rho} I_0(\rho)\rho\,d\rho \\
&= \frac{2\pi}{(w^2-1)^{3/2}}.
\end{aligned}$$

where I_0 is the modified Bessel function of the first kind. More generally,

$$\Lambda_C(u,v,w) = \frac{2\pi}{(w^2-u^2-v^2)^{3/2}}$$

for all $(u,v,w) \in \mathbb{R}^3$ such that $w > \sqrt{u^2+v^2}$.

4.3.2 Approximating integrals on a cone by discrete sums

We begin with a very standard lemma. This result is well known but we could not find a proof in the literature, so we give one.

Lemma 4.5. *Let $L > 0$, and let K be a compact convex subset of the hypercube $[-\frac{L}{2}, \frac{L}{2}]^d$. For every Lipschitz continuous function $f : K \to \mathbb{R}$ with Lipschitz constant $M > 0$,*

$$\left| \sum_{\mathbf{x} \in K \cap \mathbb{Z}^d} f(\mathbf{x}) - \int_K f(\mathbf{x})\, d\mathbf{x} \right| \leq M \frac{\sqrt{d}}{2} L^d + 4d!(L+1)^{d-1} \sup_K |f|. \tag{4.1}$$

Proof. For all $\mathbf{x}$ in $\mathbb{Z}^d$, let us consider the (hyper)cube $Q(\mathbf{x}) = \mathbf{x} + [-\frac{1}{2}, \frac{1}{2}]^d$ of unit volume. These cubes are of three types: those that are contained in K, those that cross the boundary of K, and those that have no point in common with K. The idea of the proof is to approximate the integral by considering in first approximation only the reunion of the cubes of the first kind. For each cube $Q(\mathbf{x})$ contained in K,

$$\left| f(\mathbf{x}) - \int_{Q(\mathbf{x})} f(\mathbf{y}) d\mathbf{y} \right| \leq \int_{Q(\mathbf{x})} |f(\mathbf{x}) - f(\mathbf{y})|\, d\mathbf{y} \leq M \frac{\sqrt{d}}{2}.$$

Moreover, the number of cubes contained in K is at most L^d. This already yields the first term in the right-hand side of (4.1).

We also have to deal with the cubes that cross the boundary of K. Since K is convex, it is easily seen by induction on d that the number of such cubes is at most $2d!(L+1)^{d-1}$. Bounding f by $\sup_K |f|$ on $Q(x) \cap K$, we obtain (4.1). □

Corollary 4.6. *Let $f : C \to \mathbb{R}$ be continuously differentiable and positively homogeneous of degree h, that is to say $f(\lambda \mathbf{x}) = \lambda^h f(\mathbf{x})$ for all $\mathbf{x} \in C$ and $h \geq 0$. Let A be a compact subset of $\mathcal{D}(C)$. There exists $c_{A,f,d} > 0$ such that for all $\beta > 0$,*

$$\sup_{\mathbf{u} \in A} \left| \sum_{\mathbf{x} \in C \cap \mathbb{Z}^d} f(\mathbf{x}) e^{-\beta \langle \mathbf{u}, \mathbf{x} \rangle} - \int_C f(\mathbf{x}) e^{-\beta \langle \mathbf{u}, \mathbf{x} \rangle} d\mathbf{x} \right| \leq \frac{c_{A,f,d}}{\beta^{d+h-1}}. \tag{4.2}$$

Proof. Since A is compact, we can find $L > 0$ such that for all $\mathbf{u} \in A$, the truncated cone $C_\mathbf{u} = \{x \in C : \langle \mathbf{u}, \mathbf{x} \rangle \leq 1\}$ is contained in $[-\frac{L}{2}, \frac{L}{2}]^d$. For all $t > 0$, an application of Lemma 4.5 to the compact convex subset $tC_\mathbf{u}$ of $[-\frac{tL}{2}, \frac{tL}{2}]^d$ implies

$$\sup_{\mathbf{u} \in A} \left| \sum_{\mathbf{x} \in tC_\mathbf{u} \cap \mathbb{Z}^d} f(\mathbf{x}) - \int_{tC_\mathbf{u}} f(\mathbf{x}) d\mathbf{x} \right| \ll_{f,A,d} (1+t)^{d+h-1}.$$

By integration over $[0,+\infty)$, we obtain therefore for all $\beta > 0$,

$$\sup_{\mathbf{u}\in A} \int_0^\infty \left| \sum_{\mathbf{x}\in tC_{\mathbf{u}}\cap\mathbb{Z}^d} f(\mathbf{x}) - \int_{tC_{\mathbf{u}}} f(\mathbf{x})dx \right| \beta e^{-\beta t} dt \ll_{f,A,d} \frac{1}{\beta^{d+h-1}}.$$

This concludes the proof since the Fubini theorem yields

$$\int_0^\infty \left[\sum_{\mathbf{x}\in C\cap\mathbb{Z}^d} f(\mathbf{x}) \mathbf{1}_{\{\langle \mathbf{u},\mathbf{x}\rangle \le t\}} \right] \beta e^{-\beta t} dt = \sum_{\mathbf{x}\in C\cap\mathbb{Z}^d} f(\mathbf{x}) e^{-\beta\langle \mathbf{u},\mathbf{x}\rangle}$$

and similarly

$$\int_0^\infty \left[\int_{tC_{\mathbf{u}}} f(\mathbf{x}) d\mathbf{x} \right] \beta e^{-\beta t} dt = \int_{\mathbf{x}\in C} f(\mathbf{x}) e^{-\beta\langle \mathbf{u},\mathbf{x}\rangle}.$$

□

Corollary 4.7. *Let $\mathbb{Z}_*^d = \{\mathbf{x} \in \mathbb{Z}^d : \gcd(x_1,\dots,x_d) = 1\}$ be the set of lattice points visible from the origin. Let $f : C \to \mathbb{R}$ be continuously differentiable and positively homogeneous of degree h, that is to say $f(\lambda\mathbf{x}) = \lambda^h f(\mathbf{x})$ for all $\mathbf{x} \in C$ and $h \ge 0$. Let A be a compact subset of $\mathcal{D}(C)$. There exists $c_{A,f,d} > 0$ such that for all $\beta > 0$,*

$$\sup_{\mathbf{u}\in A} \left| \sum_{\mathbf{x}\in C\cap\mathbb{Z}_*^d} f(\mathbf{x}) e^{-\beta\langle \mathbf{u},\mathbf{x}\rangle} - \frac{1}{\zeta(d)} \int_C f(\mathbf{x}) e^{-\beta\langle \mathbf{u},\mathbf{x}\rangle} dx \right| \le \frac{c_{A,f,d}}{\beta^{d+h-1}}. \tag{4.3}$$

Proof. This follows from Corollary 4.6 and the fact that the subset $\mathbb{Z}_*^d$ of $\mathbb{Z}^d$ has asymptotic density $1/\zeta(d)$, which is a well known consequence of the Möbius inversion formula [30]. □

4.4 The number of zonotopes

In this section, we prove a logarithmic estimate for the number of zonotopes in a cone with given endpoint. The main result is the following theorem.

Theorem 4.1. *Let C be a directed cone in $\mathbb{R}^d$ with Laplace transform Λ_C. For every direction $\mathbf{k} \in \mathbb{Z}^d$ in the interior of C, the number $p_C(n\mathbf{k})$ of zonotopes in C with endpoint $n\mathbf{k}$ satisfies*

$$\frac{\log p_C(n\mathbf{k})}{n^{d/(d+1)}} \xrightarrow[n\to\infty]{} \sqrt[d+1]{\frac{\zeta(d+1)}{\zeta(d)}} \min_{\mathbf{u}\in\mathbb{R}^d} [\Lambda_C(\mathbf{u}) + \langle \mathbf{u},\mathbf{k}\rangle].$$

In the next section, we start by giving two examples of application of the theorem. The following sections provide the details for the proof of the theorem, the organization of which we indicate now.

Organization of the proof. The leading idea of the proof is to introduce a new random model for zonotopes which is based on the correspondence with integer partitions. This is done in 4.4.2. For this model, the equation 4.4 on the next page shows the numbers of zonotopes $p_C(n\mathbf{k})$ can be recovered from the exact formula

$$\mathbb{P}_{\beta\mathbf{u}}[\mathbf{X} = n\mathbf{k}] = \frac{p_C(n\mathbf{k})}{Z(\beta\mathbf{u})} e^{-\beta n \langle \mathbf{u}, \mathbf{k} \rangle}$$

where $\beta > 0$ and $\mathbf{u} \in \mathbb{R}^d$ are free parameters of the model. Since we are interested only in the exponential regime, we need to chose the parameters so that the probability $\mathbb{P}_{\beta\mathbf{u}}[\mathbf{X} = n\mathbf{k}]$ is at most polynomially decreasing.

A natural guess is to aim first for $\mathbb{E}_{\beta\mathbf{u}}[\mathbf{X}] \sim n\mathbf{k}$. This can be achieved by taking $\mathbf{u}$ such that $-\nabla\Lambda_C(\mathbf{u}) = \mathbf{k}$, which corresponds exactly to the minimization problem. The choice of $\beta > 0$ as a function of n is then forced by the asymptotic estimates of Proposition 4.8.

From this point, we need to justify that $\mathbb{P}_{\beta\mathbf{u}}[\mathbf{X} = n\mathbf{k}]$ is indeed at most polynomially decreasing. We prove this fact by establishing a local central limit theorem. □

4.4.1 Examples

We first consider the case of the nonnegative orthant Orth $= [0, \infty)^d$. We have seen previously that its Laplace transform satisfies

$$\Lambda_{\mathrm{Orth}}(\mathbf{u}) = \begin{cases} \dfrac{1}{u_1 \dots u_d} & \text{if } \mathbf{u} \in (0, \infty)^d \\ +\infty & \text{otherwise} \end{cases}.$$

This makes the problem of $\mathbf{u}$ minimizing $\Lambda_C(\mathbf{u}) + \langle \mathbf{u}, \mathbf{k} \rangle$ easy to solve and we find

$$\min_{u \in \mathbb{R}^d} [\Lambda_{\mathrm{Orth}}(\mathbf{u}) + \mathbf{u} \cdot \mathbf{k}] = (d+1)(k_1 k_2 \cdots k_d)^{1/(d+1)}$$

for all $\mathbf{k} \in (0, \infty)^d$. Therefore,

$$\lim_{n \to \infty} \frac{\log p_{\mathrm{Orth}}(n\mathbf{k})}{n^{d/(d+1)}} = (d+1) \sqrt[d+1]{\frac{\zeta(d+1)}{\zeta(d)} (k_1 k_2 \cdots k_d)}.$$

In the case $d = 2$ and $\mathbf{k} = (1, 1)$, this recovers the result of [53, 8, 44].

Let us now consider the example of the Euclidean cone C in $\mathbb{R}^3$ of equation $x^2 + y^2 \le z^2$ with $z \ge 0$. Its Laplace transform has been computed earlier:

$$\Lambda_C(u, v, w) = \begin{cases} \dfrac{2\pi}{(w^2 - u^2 - v^2)^{3/2}}, & \text{if } w^2 > u^2 + v^2 \\ +\infty & \text{otherwise} \end{cases}$$

Some basic multivariate calculus leads to

$$\min_{\mathbf{u}\in\mathbb{R}^d}[\Lambda_C(\mathbf{u})+\langle\mathbf{u},\mathbf{k}\rangle] = 4\sqrt[4]{\frac{2\pi}{27}}(|k_3|^2-|k_1|^2-|k_2|^2)^{3/8}$$

for all $\mathbf{k}$ in the interior of C. It follows that for $\mathbf{k}=(0,0,1)$, the conclusion of the theorem can be written as

$$\lim_{n\to\infty}\frac{\log p_C(0,0,n)}{n^{3/4}} = 4\sqrt[4]{\frac{2\pi\zeta(4)}{27\zeta(3)}}.$$

4.4.2 Random models for integer partitions and zonotopes

We define a probabilistic model of strict integer partitions and zonotopes which is similar to the one introduced by Sinaĭ in [44]. Remember from Section 4.2 the vector-valued function $\mathbf{X}$ defined on the set Ω of all multiplicity functions $\omega\colon \mathbb{Z}^d_* \cap C \to \mathbb{Z}_+$ with finite support by the summation

$$\mathbf{X}(\omega) := \sum_{\mathbf{x}\in\mathbb{Z}^d_*\cap C} \omega(\mathbf{x})\mathbf{x}.$$

We observed that a multiplicity function ω describes a strict partition of $\mathbf{n}$ if and only if the condition $\mathbf{X}(\omega)=\mathbf{n}$ is met. We use this additive structure to build a model of random partitions satisfying the two following properties:

- The random variables $(\omega(\mathbf{x}))_{\mathbf{x}\in\mathbb{Z}^d_*}$ are mutually independent.
- Conditional on the event $\{\mathbf{X}=\mathbf{n}\}$, the distribution of ω is uniform among all partitions of $\mathbf{n}$.

This is achieved by the following construction. Let $\mathbb{Z}^d_C = \mathbb{Z}^d_* \cap C\setminus\{0\}$ be the set of non-zero lattice vectors lying in the cone C, and let Ω be the set of all functions $\omega\colon \mathbb{Z}^d_C \to \mathbb{Z}_+$ with finite support. For all parameter $\mathbf{u}\in\mathbb{R}^d$, let

$$\mathbb{P}_{\mathbf{u}}(\omega) := \frac{1}{Z(\mathbf{u})}\exp[-\langle\mathbf{u},\mathbf{X}(\omega)\rangle],$$

which is well defined for every $\mathbf{u}$ such that the so-called *partition function*

$$Z(\mathbf{u}) := \sum_{\omega\in\Omega} e^{-\langle\mathbf{u},\mathbf{X}(\omega)\rangle}$$

is convergent. It does for every $\mathbf{u}\in\mathcal{D}(C)$. The distribution of ω is uniform conditional on the event $\mathbf{X}=\mathbf{n}$ because for every ω such that $\mathbf{X}(\omega)=\mathbf{n}$, one has

$$\mathbb{P}_{\mathbf{u}}(\omega) = \frac{1}{Z(\mathbf{u})}e^{-\langle\mathbf{u},\mathbf{n}\rangle}, \tag{4.4}$$

where the right-hand side depends only of $\mathbf{n}$.

It is a well known fact that, for probability measures of this form, the distribution of the random vector $\mathbf{X}$ under the measure $\mathbb{P}_{\mathbf{u}}$ can be fully recovered from the partition function $Z(\mathbf{u})$. For example, the expected value of $\mathbf{X}$ is

$$\mathbb{E}_{\mathbf{u}}[\mathbf{X}] = \sum_{\omega\in\Omega} \mathbf{X}(\omega)\frac{e^{-\langle \mathbf{u},\mathbf{X}(\omega)\rangle}}{Z(\mathbf{u})} = -\frac{1}{Z(\mathbf{u})}\nabla Z(\mathbf{u}) = -\nabla(\log Z)(\mathbf{u}).$$

More generally, $\log Z$ is a generating function for all *cumulants* of the distribution of $\mathbf{X}$. Cumulants include the covariance matrix which is obtained from the Hessian matrix $\mathrm{Hess}(\log Z)(\mathbf{u})$.

Since our strategy will be to calibrate $\mathbf{u}$ such that $\mathbb{E}_{\mathbf{u}}[\mathbf{X}] = \mathbf{n}$, we need a good estimate for $\log Z$. Because of the multiplicative structure of the probability measure $\mathbb{P}_{\mathbf{u}}$, the family of random variables $(\omega(\mathbf{x}))_{\mathbf{x}\in\mathbb{Z}^d_C}$ is independent. In addition, from the definition of $\mathbb{P}_{\mathbf{u}}$, one can see that for all $\mathbf{x}\in\mathbb{Z}^d_C$, the distribution of the integer-valued random variable $\omega(\mathbf{x})$ under $\mathbb{P}_{\mathbf{u}}$ is geometric with parameter $1-e^{-\langle \mathbf{u},\mathbf{x}\rangle}$, that is to say

$$\mathbb{P}_{\mathbf{u}}[\omega(\mathbf{x}) = k] = (1-e^{-\langle \mathbf{u},\mathbf{x}\rangle})\, e^{-k\langle \mathbf{u},\mathbf{x}\rangle}, \qquad k\in\mathbb{Z}_+.$$

These two facts lead to a remarkable factorization of $Z(\mathbf{u})$ as convergent product which is similar to Euler's formula,

$$Z(\mathbf{u}) = \prod_{\mathbf{x}\in\mathbb{Z}^d_C} \frac{1}{1-e^{-\langle \mathbf{u},\mathbf{x}\rangle}}.$$

Taking logarithms, we obtain finally a series expansion for $\log Z$,

$$\log Z(\mathbf{u}) = \sum_{\mathbf{x}\in\mathbb{Z}^d_C}\sum_{r\geq 1} \frac{e^{-r\langle \mathbf{u},\mathbf{x}\rangle}}{r}. \tag{4.5}$$

This expression is very convenient in order to obtain some asymptotic estimates by doing comparisons with integrals. The next proposition is a reformulation of Corollary 4.7 on page 86.

Proposition 4.8. *Let A be a compact subset of $\mathcal{D}(C)$. Let Λ_C be the Laplace transform of C.*

$$\exists c>0, \forall \beta>0, \qquad \sup_{\mathbf{u}\in A}\left|\log Z(\beta\mathbf{u}) - \frac{\zeta(d+1)\Lambda_C(\mathbf{u})}{\zeta(d)\beta^d}\right| \leq \frac{c}{\beta^{d-1}}$$

Let $m_{\beta\mathbf{u}}$ denote the expectation of $\mathbf{X}$ under the distribution $\mathbb{P}_{\beta\mathbf{u}}$.

$$\exists c>0, \forall \beta>0, \qquad \sup_{\mathbf{u}\in A}\left|-m_{\beta\mathbf{u}} - \frac{\zeta(d+1)\nabla\Lambda_C(\mathbf{u})}{\zeta(d)\beta^{d+1}}\right| \leq \frac{c}{\beta^d}$$

Let $\Gamma_{\beta\mathbf{u}}$ denote the covariance matrix of $\mathbf{X}$ *under the distribution* $\mathbb{P}_{\beta\mathbf{u}}$.

$$\exists c > 0, \forall \beta > 0, \qquad \sup_{\mathbf{u}\in A} \left| \Gamma_{\beta\mathbf{u}} - \frac{\zeta(d+1)\,\mathrm{Hess}\,\Lambda_C(\mathbf{u})}{\zeta(d)\beta^{d+2}} \right| \le \frac{c}{\beta^{d+1}}$$

4.4.3 Calibration of the parameter

In view of Theorem 4.1, we need to choose the parameter $\mathbf{u} \in \mathbb{R}^d$ of the probability measure $\mathbb{P}_{\mathbf{u}}$ in order that $\mathbb{P}_{\mathbf{u}}[\mathbf{X} = n\mathbf{k}]$ is not too small. A natural guess is to aim first for $\mathbb{E}_{\mathbf{u}}[\mathbf{X}] \sim n\mathbf{k}$. It turns out that this can be done by considering scalings of a unique vector $\mathbf{u}$ in $\mathcal{D}(C)$. Indeed, Proposition 4.8 on the previous page shows that if we define β and $\mathbf{u}$ by the equations

$$-\nabla\Lambda_C(\mathbf{u}) = \mathbf{k}, \tag{4.6}$$

$$\beta^{d+1} n = \frac{\zeta(d+1)}{\zeta(d)}, \tag{4.7}$$

then the following estimate is automatically satisfied

$$\frac{1}{n}\,\mathbb{E}_{\beta\mathbf{u}}[\mathbf{X}] \xrightarrow[n\to\infty]{} \mathbf{k}.$$

From now on, we will always assume that the equations (4.6) and (4.7) are satisfied.

Only the existence of a solution $\mathbf{u}$ to (4.6) needs to be justified. This can be done by considering the minimization problem

$$\inf_{\mathbf{u}\in\mathbb{R}^d} [\Lambda_C(\mathbf{u}) + \langle \mathbf{u}, \mathbf{k}\rangle].$$

The function Λ_C is strictly convex, smooth, and goes to $+\infty$ on any point of $\partial\mathcal{D}(C)$. So there exists a vector $\mathbf{u} \in \mathcal{D}(C)$ which minimizes this functional. Since $\mathbf{u}$ must be a critical point of the functional, equation (4.6) follows.

Actually, this calibration turns out to be too loose and we need a more precise one: choosing β and $\mathbf{u}$ such that $\mathbb{E}_{\beta\mathbf{u}}[\mathbf{X}] = n\mathbf{k}$ instead of $\mathbb{E}_{\beta\mathbf{u}}[\mathbf{X}] \sim n\mathbf{k}$. The price we pay for this is that equations 4.6 and 4.7 are not strictly valid anymore, but still hold if understood as asymptotic equivalents.

4.4.4 Local limit theorem

In order to finish the proof of Theorem 4.1, we need to show that $\log \mathbb{P}_{\beta\mathbf{u}}[\mathbf{X} = n\mathbf{k}]$ does not compete with $n^{d/d+1}$ when the parameters β and $\mathbf{u}$ have been chosen according to the equations (4.6) and (4.7).

Proposition 4.9.

$$\mathbb{P}_{\beta\mathbf{u}}[\mathbf{X} = n\mathbf{k}] = e^{-O(\log n)}$$

Proof. Actually, this result will be a consequence of a much stronger statement. For every $\mathbf{u} \in \mathcal{D}(C)$, a Gaussian local limit theorem holds for the distribution of $\mathbf{X}$ under the measure $\mathbb{P}_{\beta\mathbf{u}}$ in the following sense. Let $\mathbf{m}_{\beta\mathbf{u}}$ and $\Gamma_{\beta\mathbf{u}}$ denote respectively the mean of $\mathbf{X}$ and the covariance quadratic form of $\mathbf{X}$. One has

$$\sup_{\mathbf{n}\in\mathbb{Z}^d}\left|\mathbb{P}_{\beta\mathbf{u}}[\mathbf{X}=\mathbf{n}] - \frac{e^{-\frac{1}{2}\Gamma_{\beta\mathbf{u}}^{-1}(\mathbf{n}-\mathbf{m}_{\beta\mathbf{u}})}}{(2\pi)^{d/2}\sqrt{\det\Gamma_{\beta\mathbf{u}}}}\right| = O(\beta^{d(d+3)/2})$$

We obtain this result by applying Proposition $7\cdot 1$ from [19]. Let us see why this implies Proposition 4.9, notice that Proposition 4.8 implies for $\beta \to 0$

$$\Gamma_{\beta\mathbf{u}} = \frac{H(\mathbf{u})}{\beta^{d+2}} + O(\beta^{-(d+1)})$$

where $H(\mathbf{u}) = c\,\mathrm{Hess}(\log\Lambda_C)(\mathbf{u})$ is non singular. In particular, $\det(\Gamma_{\beta\mathbf{u}}) \asymp \beta^{d(d+2)}$, so

$$\log\mathbb{P}_{\beta\mu}[X = n\mathbf{k}] = -\frac{\beta^{d+2}}{2}H(\mathbf{u})^{-1}(n\mathbf{k}-\mathbf{m}_{\beta\mathbf{u}}) + O(\log\beta).$$

Finally, recall the equation $\mathbf{m}_{\beta\mathbf{u}} = \mathbb{E}_{\beta\mathbf{u}}[\mathbf{X}] = n\mathbf{k}$ coming from the calibration, which cancels the first term.

The application of Proposition $7\cdot 1$ from [19] requires to check three conditions on the model. The condition about the smallest eigenvalue of $\Gamma_{\beta\mathbf{u}}$ follows from the estimate above. The two remaining conditions, which ask for bounds on the Lyapunov ratio and on the characteristic function, are the subjects of the next two propositions. □

Proposition 4.10 (bound on the Lyapunov ratio)**.** *For all* $\mathbf{x} \in \mathbb{Z}_C^d$*, let* $\overline{\mathbf{X}_\mathbf{x}}(\omega) = (\omega(\mathbf{x}) - \mathbb{E}_{\beta\mathbf{u}}[\omega(\mathbf{x})])\mathbf{x}$*. There exists a constant* $c > 0$ *such that for all* $\beta > 0$

$$L_{\beta\mathbf{u}} = \sup_{\mathbf{v}\in\mathbb{R}^d\setminus\{0\}}\sum_{\mathbf{x}\in\mathbb{Z}_C^d}\frac{\mathbb{E}_{\beta\mathbf{u}}|\langle\mathbf{v},\overline{\mathbf{X}_\mathbf{x}}\rangle|^3}{\|\Gamma_{\beta\mu}^{1/2}\mathbf{v}\|^3} \le \|\Gamma_{\beta\mathbf{u}}^{-1/2}\|^3\sum_{\mathbf{x}\in\mathbb{Z}_C^d}\mathbb{E}_{\beta\mathbf{u}}\left(\|\overline{\mathbf{X}_\mathbf{x}}\|^3\right) \le \beta^{d/2}$$

Proof. The first inequality is justified by a change of variable $v = \Gamma_{\beta\mathbf{u}}^{-1/2}v'$ and the Cauchy-Schwarz inequality. Applying the Cauchy-Schwarz inequality again yields

$$\sum_{\mathbf{x}\in\mathbb{Z}_C^d}\mathbb{E}_{\beta\mathbf{u}}\left(\|\overline{X_\mathbf{x}}\|^3\right) \le \sum_{\mathbf{x}\in\mathbb{Z}_C^d}\sqrt{\mathbb{E}_{\beta\mathbf{u}}\left(\|\overline{X_\mathbf{x}}\|^2\right)\mathbb{E}_{\beta\mathbf{u}}\left(\|\overline{X_\mathbf{x}}\|^4\right)}$$

Since $\omega(\mathbf{x})$ has geometric distribution with probability of failure $e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle}$,

$$\mathbb{E}_{\beta\mathbf{u}}\left(\|\overline{X_\mathbf{x}}\|^2\right) = \|\mathbf{x}\|^2\,\mathrm{Var}_{\beta\mathbf{u}}(\omega(\mathbf{x})) = \frac{\|\mathbf{x}\|^2 e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle}}{(1-e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle})^2}$$

and similarly

$$\mathbb{E}_{\beta\mathbf{u}}\left(\|\overline{X_{\mathbf{x}}}\|^4\right) = \|\mathbf{x}\|^4 \frac{e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle}(1+7e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle}+e^{-2\beta\langle\mathbf{u},\mathbf{x}\rangle})}{(1-e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle})^4} \leq \frac{9\|\mathbf{x}\|^4 e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle}}{(1-e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle})^4}.$$

Therefore,

$$\sum_{\mathbf{x}\in\mathbb{Z}_C^d} \mathbb{E}_{\beta\mathbf{u}}\left(\|\overline{X_{\mathbf{x}}}\|^3\right) \leq 3 \sum_{\mathbf{x}\in\mathbb{Z}_C^d} \frac{\|\mathbf{x}\|^3 e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle}}{(1-e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle})^3}.$$

Now, we use Corollary 4.7 to conclude. □

Proposition 4.11 (bound on the characteristic function)**.** *Let $\mathcal{E}_{\beta\mathbf{u}}$ be the ellipsoid of $\mathbb{R}^d$ defined by*

$$\mathcal{E}_{\beta\mathbf{u}} = \left\{\mathbf{v}\in\mathbb{R}^d : \Gamma_{\beta\mathbf{u}}(\mathbf{v},\mathbf{v}) \leq \frac{1}{16L_{\beta\mathbf{u}}^2}\right\}.$$

There exists a constant $c>0$ such that for all $\beta>0$,

$$\sup_{\mathbf{v}\in[-\pi,\pi]^d\setminus\mathcal{E}_\beta} \left|\mathbb{E}_{\beta\mathbf{u}}\left(e^{i\langle\mathbf{v},\mathbf{X}\rangle}\right)\right| \leq \exp\left(-\frac{c}{\beta^d}\right).$$

Proof. Let us introduce the notation $\phi_{\beta\mathbf{u}}(\mathbf{v}) = \mathbb{E}_{\beta\mathbf{u}}\left(e^{i\langle\mathbf{v},\mathbf{X}\rangle}\right)$. A straightforward computation yields

$$|\phi_{\beta\mathbf{u}}(\mathbf{v})|^2 = \prod_{\mathbf{x}\in\mathbb{Z}_C^d} \frac{(1-e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle})^2}{(1-e^{-\langle\beta\mathbf{u}-i\mathbf{v},\mathbf{x}\rangle})(1-e^{-\langle\beta\mathbf{u}+i\mathbf{v},\mathbf{x}\rangle})},$$

hence

$$\begin{aligned}\log|\phi_{\beta\mathbf{u}}(\mathbf{v})| &= -\sum_{\mathbf{x}\in\mathbb{Z}_C^d}\sum_{k\geq1} \frac{e^{-k\beta\langle\mathbf{u},\mathbf{x}\rangle}}{k}(1-\cos(k\langle\mathbf{v},\mathbf{x}\rangle)) \\ &\leq -\sum_{\mathbf{x}\in\mathbb{Z}_C^d} e^{-\beta\langle\mathbf{u},\mathbf{x}\rangle}(1-\cos(\langle\mathbf{v},\mathbf{x}\rangle))\end{aligned}$$

The remaining of proof is similar to the proof of Lemma 3.9 on page 60. □

4.5 Limit shapes

In this section, we are interested in the existence of a limit shape for random zonotopes drawn under the uniform distribution on the set of all lattice zonotopes in the cone C with endpoint $\mathbf{n}$.

Before we state the main theorem, let us recall some well known facts from convex geometry. The set of all convex compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff distance

$$(A,B) \mapsto d_H(A,B) = \max\left(\sup_{\mathbf{u}\in A}\inf_{\mathbf{v}\in B}\|\mathbf{u}-\mathbf{v}\|, \sup_{\mathbf{v}\in B}\inf_{\mathbf{u}\in A}\|\mathbf{u}-\mathbf{v}\|\right)$$

is a complete and separable metric space. For every convex compact subset A of $\mathbb{R}^d$, the *support function* of A is the continuous function $h_A\colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$h_A(\mathbf{w}) = \sup_{\mathbf{v}\in A}\langle \mathbf{v},\mathbf{w}\rangle, \qquad \mathbf{w}\in\mathbb{R}^d.$$

The support function of A characterizes A. To be more precise, a theorem of Minkowski shows that for any convex compact subsets A and B of $\mathbb{R}^d$,

$$d_H(A,B) = \sup_{\mathbf{w}\in\mathbb{S}^{d-1}} |h_A(\mathbf{w}) - h_B(\mathbf{w})|.$$

In other words, the Minkowski function $M\colon A \to h_A(\cdot)$ induces an isometric mapping between the set of convex compact subsets of $\mathbb{R}^d$ and the Banach space of continuous functions on $\mathbb{S}^{d-1}$ endowed with the uniform distance.

If A and B are convex compact subsets of $\mathbb{R}^d$, their Minkowski sum is the convex compact subset $A\oplus B = \{\mathbf{u}+\mathbf{v}; \mathbf{u}\in A, \mathbf{v}\in B\}$. In terms of support functions, $h_{A\oplus B} = h_A + h_B$. Yet another property will be useful: if A is a convex compact, then $\lambda A = \{\lambda\mathbf{u}; \mathbf{u}\in A\}$ satisfies $h_{\lambda A} = \lambda h_A$ if $\lambda \geq 0$.

The following theorem expresses the existence of a limit shape in terms of the support function.

Theorem 4.2. *Let C be a convex cone and let $\mathbf{k}\in\mathbb{Z}^d$ be interior to C. Let* $\mathrm{Zono}(C, n\mathbf{k})$ *denote the set of all lattice zonotopes in C with ending point $n\mathbf{k}$. Let $\mathbf{u}$ be the unique solution to the minimization problem*

$$\min_{u\in\mathbb{R}^d}\,[\Lambda_C(\mathbf{u}) + \langle \mathbf{u},\mathbf{k}\rangle].$$

For every $\mathbf{v}\in\mathbb{R}^d$, let $\mathcal{H}_\mathbf{v}$ be the half-space of all vectors $\mathbf{x}\in\mathbb{R}^d$ with $\langle\mathbf{x},\mathbf{v}\rangle \geq 0$ and let

$$h(\mathbf{v}) = \left\langle -\nabla\Lambda_{C\cap\mathcal{H}_\mathbf{v}}(\mathbf{u}), \mathbf{v}\right\rangle.$$

Then, for all $\mathbf{v}\in\mathbb{R}^d$ and for all $\varepsilon > 0$,

$$\lim_{n\to+\infty}\frac{\#\left\{\mathcal{Z}\in\mathrm{Zono}(C,n\mathbf{k}) : |h_{\frac{1}{n}\mathcal{Z}}(\mathbf{v}) - h(\mathbf{v})| > \varepsilon\right\}}{p_C(n\mathbf{k})} = 0.$$

Proof. Again, the proof of this theorem is based on the probabilistic model of Section 4.4.2. It depends on the fact that, conditional on the event $\mathbf{X} = n\mathbf{k}$, the distribution of a random zonotope under $\mathbb{P}_{\beta\mathbf{u}}$ is uniformly distributed on the set $\mathrm{Zono}(C, n\mathbf{k})$. In particular,

$$\frac{\#\{\mathcal{Z} \in \mathrm{Zono}(C, n\mathbf{k}) : |h_{\frac{1}{n}\mathcal{Z}}(\mathbf{v}) - h(\mathbf{v})| > \varepsilon\}}{p_C(n\mathbf{k})} = \mathbb{P}_{\beta\mathbf{u}}\left[|h_{\frac{1}{n}\mathcal{Z}}(\mathbf{v}) - h(\mathbf{v})| > \varepsilon \mid \mathbf{X} = n\mathbf{k}\right]$$

$$\leq \frac{\mathbb{P}_{\beta\mathbf{u}}\left[|h_{\frac{1}{n}\mathcal{Z}}(\mathbf{v}) - h(\mathbf{v})| > \varepsilon\right]}{\mathbb{P}_{\beta\mathbf{u}}[\mathbf{X} = n\mathbf{k}]}.$$

Hence, we need only prove that the right-hand side of this inequality goes to 0. Using Proposition 4.9, we see that the theorem follows from the result of the next Proposition. □

The following proposition, which is instrumental in proving the theorem, establishes the existence of the limit shape under the measure $\mathbb{P}_{\beta\mathbf{u}}$. It is actually stronger than that since it also provides large deviation bounds from this shape.

Proposition 4.12. *Let C be a convex cone and let $\mathbf{k} \in \mathbb{Z}^d$ be interior to C. Let* $\mathrm{Zono}(C, n\mathbf{k})$ *denote the set of all lattice zonotopes in C with ending point $n\mathbf{k}$. Let $\mathbf{u}$ be the unique solution to the minimization problem*

$$\min_{u \in \mathbb{R}^d} [\Lambda_C(\mathbf{u}) + \langle \mathbf{u}, \mathbf{k} \rangle].$$

For every $\mathbf{v} \in \mathbb{R}^d$, let $\mathcal{H}_\mathbf{v}$ be the half-space of all vectors $\mathbf{x} \in \mathbb{R}^d$ with $\langle \mathbf{x}, \mathbf{v} \rangle \geq 0$ and let

$$h(\mathbf{v}) = \left\langle -\nabla \Lambda_{C \cap \mathcal{H}_\mathbf{v}}(\mathbf{u}), \mathbf{v} \right\rangle.$$

Then, for all $\mathbf{v} \in \mathbb{R}^d$ and for all $\varepsilon > 0$, there exists $c > 0$ such that for all n large enough,

$$\mathbb{P}_{\beta\mathbf{u}}\left[|h_{\frac{1}{n}\mathcal{Z}}(\mathbf{v}) - h(\mathbf{v})| > \varepsilon\right] \leq e^{-c\, n^{d/(d+1)}}.$$

Proof. Let us first remark that for a zonotope $\mathcal{Z}$ generated by vectors $\mathbf{w}^1, \ldots, \mathbf{w}^k$,

$$h_\mathcal{Z}(\mathbf{v}) = \sup_{(\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k} \langle \sum_{i=1}^k \varepsilon_i \mathbf{w}^i, \mathbf{v} \rangle = \sup_{(\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k} \sum_{i=1}^k \varepsilon_i \langle \mathbf{w}^i, \mathbf{v} \rangle$$
$$= \sum_{i=1}^k \langle \mathbf{w}^i, \mathbf{v} \rangle \mathbf{1}_{\{\langle \mathbf{w}^i, \mathbf{v} \rangle \geq 0\}}$$

Therefore, the zonotope $\mathcal{Z}$ generated by the vectors $\mathbf{x} \in \mathbb{Z}_C^d$ with multiplicities $\omega(\mathbf{x})$ satisfies

$$h_\mathcal{Z}(\mathbf{v}) = \sum_{\mathbf{x} \in \mathbb{Z}_C^d} \omega(\mathbf{x}) \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{1}_{\{\langle \mathbf{x}, \mathbf{v} \rangle \geq 0\}} = \sum_{\mathbf{x} \in \mathbb{Z}_C^d \cap \mathcal{H}_\mathbf{v}} \omega(\mathbf{x}) \langle \mathbf{x}, \mathbf{v} \rangle.$$

Of course, $\mathbb{Z}_C^d \cap \mathcal{H}_\mathbf{v} = \mathbb{Z}_{C \cap \mathcal{H}_\mathbf{v}}^d$. Hence we can apply Proposition 4.8 with the smaller cone

$C \cap \mathcal{H}_{\mathbf{v}}$. As $n \to \infty$, we obtain

$$\frac{1}{n}\mathbb{E}_{\beta\mathbf{u}}[h_{\mathcal{Z}}(\mathbf{v})] = \left\langle \Lambda_{C\cap\mathcal{H}_{\mathbf{v}}}(\mathbf{u}), \mathbf{v} \right\rangle + O(n^{-1/(d+1)}) = h(\mathbf{v}) + o(1).$$

Since $\frac{1}{n}h_{\mathcal{Z}} = h_{\frac{1}{n}\mathcal{Z}}$, this already shows that the limit shape appears in expectation.

We will now bound the probability for a large deviation from the mean using the so-called Chernoff method. Let us consider the exponential generating function of $h_{\mathcal{Z}}(\mathbf{v})$

$$\mathbb{E}_{\beta\mathbf{u}}[e^{\theta h_{\mathcal{Z}}(\mathbf{v})}] = \prod_{x\in\mathbb{Z}^d_{C\cap\mathcal{H}_{\mathbf{v}}}} \frac{1-e^{-\langle \beta\mathbf{u},\mathbf{x}\rangle}}{1-e^{-\langle \beta\mathbf{u}-\theta\mathbf{v},\mathbf{x}\rangle}} = \frac{Z_{C\cap\mathcal{H}_{\mathbf{v}}}(\beta\mathbf{u}-\theta\mathbf{v})}{Z_{C\cap\mathcal{H}_{\mathbf{v}}}(\beta\mathbf{u})}$$

which is well defined for all θ small enough with respect to β. Consider now the centered random variable $Y = h_{\frac{1}{n}\mathcal{Z}}(\mathbf{v}) - \mathbb{E}_{\beta\mathbf{u}}[h_{\frac{1}{n}\mathcal{Z}}(\mathbf{v})]$. From Proposition 4.8 and a second order Taylor approximation, we obtain for some constant $c(\mathbf{u},\mathbf{v}) > 0$ involving the Hessian matrix of $\Lambda_{C\cap\mathcal{H}_{\mathbf{v}}}$ at $\mathbf{u}$,

$$\log \mathbb{E}_{\beta\mathbf{u}}[e^{\theta Y}] \sim c(\mathbf{u},\mathbf{v})\,\frac{\theta^2}{2}\, n^{-d/(d+1)}$$

as long as $\frac{\theta}{n}$ goes to 0. But for all $\theta > 0$, the Markov inequality yields

$$\mathbb{P}_{\beta\mathbf{u}}[Y > \varepsilon] = \mathbb{P}_{\beta\mathbf{u}}[\theta Y \geq \theta\varepsilon] \leq e^{-\theta\varepsilon}\mathbb{E}_{\beta\mathbf{u}}[e^{\theta Y}]$$

This bound is approximately optimized for $\theta = c(\mathbf{u},\mathbf{v})^{-1}\,\varepsilon\, n^{d/(d+1)}$ and it leads to

$$\mathbb{P}_{\beta\mathbf{u}}[Y > \varepsilon] \leq \exp\left(-\frac{1}{2}c(\mathbf{u},\mathbf{v})^{-1}\varepsilon^2 n^{d/(d+1)}(1+o(1))\right)$$

A similar bound holds for $\mathbb{P}_{\beta\mathbf{u}}[-Y > \varepsilon]$, hence for $\mathbb{P}_{\beta\mathbf{u}}[|Y| > \varepsilon]$. The conclusion follows. □

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Table des matières

MÉTHODES PROBABILISTES POUR L'ÉTUDE ASYMPTOTIQUE DES PARTITIONS ENTIÈRES ET DE LA GÉOMÉTRIE CONVEXE DISCRÈTE

Résumé

Cette thèse se compose de plusieurs travaux portant sur l'énumération et le comportement asymptotique de structures combinatoires apparentées aux partitions d'entiers.
Un premier travail s'intéresse aux partitions d'entiers bipartites, qui constituent une généralisation bidimensionnelle des partitions d'entiers. Des équivalents du nombre de partitions sont obtenus dans le régime critique où l'un des entiers est de l'ordre du carré de l'autre entier et au delà de ce régime critique. Ceci complète les résultats établis dans les années cinquante par Auluck, Nanda et Wright.
Le deuxième travail traite des chaînes polygonales à sommets entiers dans le plan. Pour un modèle statistique introduit par Sinaï, une représentation intégrale exacte de la fonction de partition est donnée. Ceci conduit à un équivalent du nombre de chaînes joignant deux points distants qui fait intervenir les zéros non triviaux de la fonction zêta de Riemann. Une analyse combinatoire détaillée des chaînes convexes est présentée. Elle permet de montrer l'existence d'une forme limite pour les chaînes convexes aléatoires ayant peu de sommets, répondant ainsi à une question ouverte de Vershik.
Un troisième travail porte sur les zonotopes à sommets entiers en dimension supérieure. Un équivalent simple est donné pour le logarithme du nombre de zonotopes contenus dans un cône convexe et dont les extrémités sont fixées. Une loi des grands nombres est établie et la forme limite est caractérisée par la transformée de Laplace du cône.

Mots clés : partitions d'entiers multipartites, polygones convexes, polytopes à sommets entiers, zonotopes, théorème central limite local, combinatoire analytique

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PROBABILISTIC METHODS FOR THE ASYMPTOTIC STUDY OF INTEGRAL PARTITIONS AND DISCRETE CONVEX GEOMETRY

Abstract

This thesis consists of several works dealing with the enumeration and the asymptotic behaviour of combinatorial structures related to integer partitions.
A first work concerns partitions of large bipartite integers, which are a bidimensional generalization of integer partitions. Asymptotic formulæ are obtained in the critical regime where one of the numbers is of the order of magnitude of the square of the other number, and beyond this critical regime. This completes the results established in the fifties by Auluck, Nanda, and Wright.
The second work deals with lattice convex chains in the plane. In a statistical model introduced by Sinaï, an exact integral representation of the partition function is given. This leads to an asymptotic formula for the number of chains joining two distant points, which involves the non trivial zeros of the Riemann zeta function. A detailed combinatorial analysis of convex chains is presented. It makes it possible to prove the existence of a limit shape for random convex chains with few vertices, answering an open question of Vershik.
A third work focuses on lattice zonotopes in higher dimensions. An asymptotic equality is given for the logarithm of the number of zonotopes contained in a convex cone and such that the endings of the zonotope are fixed. A law of large numbers is established and the limit shape is characterized by the Laplace transform of the cone.

Keywords: partitions of multi-partite numbers, convex polygons, lattice polytopes, zonotopes, local central limit theorem, analytic combinatorics

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